Some properties of the Thue-Morse word
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The Thue-Morse word, also known as the Prouhet-Thue-Morse sequence, was first considered by Prouhet in 1851. This is an infinite word on the alphabet \{0, 1\}. There are many different (and yet equivalent) ways to define the Thue-Morse word. One of them is using morphisms between alphabets with the operation of concatenation, i.e., applications \( f : A \to B \) such that, for all finite words \( u, v \) on \( A \), \( f(uv) = f(u)f(v) \) and \( f(\varepsilon) = \varepsilon \) where \( \varepsilon \) denotes the empty word.

The Thue-Morse word \( t \) is the fixed point \( \lim_{n \to \infty} \varphi^n(0) \) of the morphism \( \varphi : \{0, 1\} \to \{0, 1\} \) defined by \( \varphi(0) = 01 \), \( \varphi(1) = 10 \). Hence,

\[
t = 011010011001011010011001101001 \ldots
\]

Despite the simplicity of the terms in the sequence, its properties are anything but trivial. For example, it is overlap-free. This means that there are no factor of the type \( auaua \), with \( a \in A \) and \( u \) a finite word on \( A \), appearing in the Thue-Morse word.

We will consider different complexity function of the Thue-Morse word
- factor complexity: the function counting the numbers of distinct factors,
- abelian complexity: the function counting the numbers of distinct factors up to a permutation of letters,
- \( k \)-abelian complexity: a generalization of the abelian complexity.

In particular, we will concentrate on the 2-abelian complexity of \( t \) and conjecture it is “regular” in some sense.