Invariant Games

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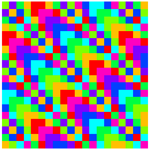
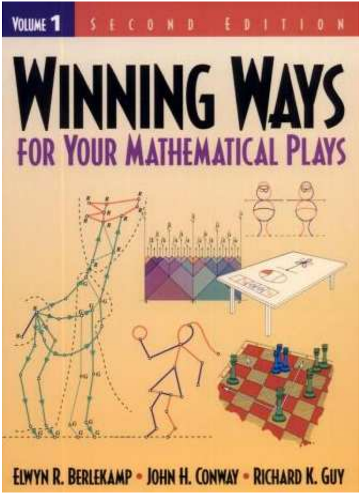


E. R. Berlekamp, J. H. Conway, R. K. Guy, Winning Ways for Your Mathematical Plays, vol. 1–4, A K Peters, Ltd (2001).



What is a (combinatorial) game ?

- There are just **two players**.
- There are several, usually finitely many, positions, and often a particular starting position.
- There are **clearly defined rules** that specify the moves that either player can make from a given position (options).
- The two players play **alternatively**.
- Both players know what is going on (**complete information**).
- There are **no chance moves**.
- In the **normal play convention** a player unable to move loses.
- The rules are such that play will always come to an end because some player will be unable to move (**ending condition**).

WYTHOFF'S GAME (1907)

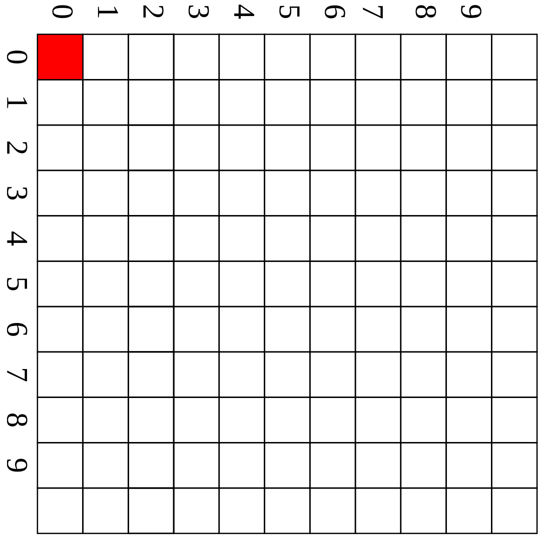
- Two players play alternatively
- Two piles of tokens
- Remove
  - any positive number of tokens from one pile (**Nim rule**) or,
  - the same positive number from the two piles.
- The one who takes the last token wins the game (last move wins).

Set of moves:

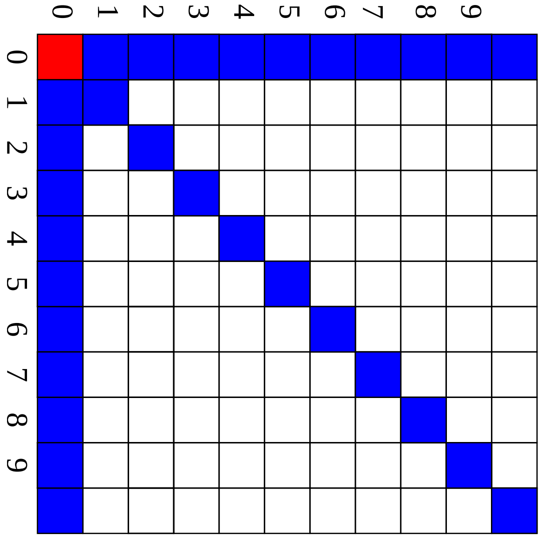
$$\mathcal{M}_W = \{(i,0),\ i>0\} \cup \{(0,j),\ j>0\} \cup \{(k,k),\ k>0\}$$

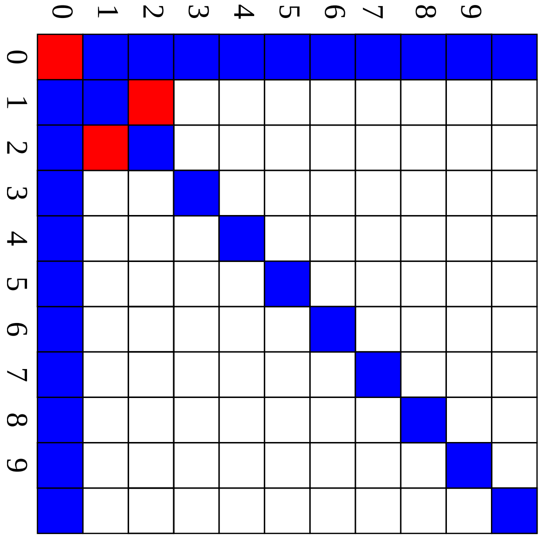
	0	1	2	3	4	5	6	7	8	9	
0											
1											
2											
3											
4											
5											
6											
7											
8											
9											

Combinatorial game theory for word combinatorists



Combinatorial game theory for word combinatorists

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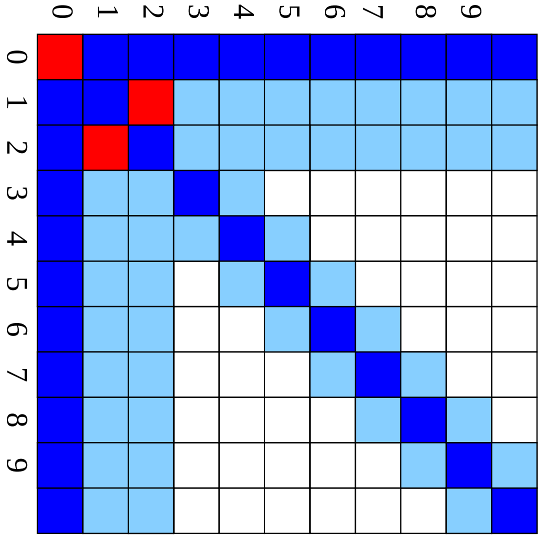
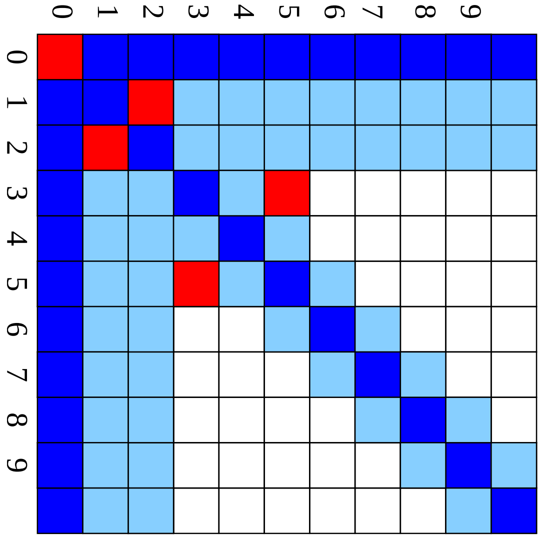
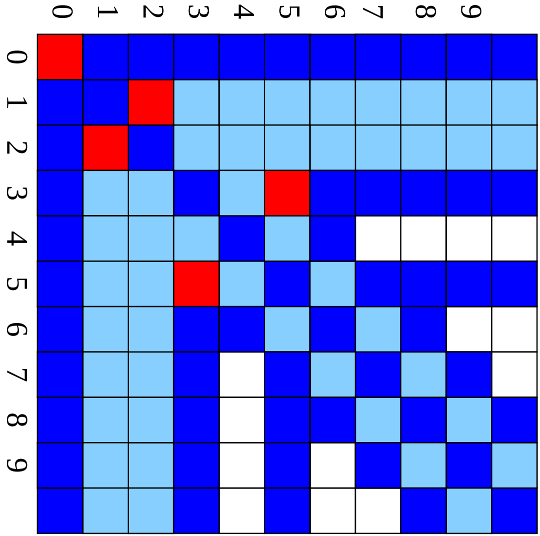
0 1 2 3 4 5 6 7 8 9

Combinatorial game theory for word combinatorists



COMBINATORIAL GAME THEORY FOR WORD COMBINATORISTS

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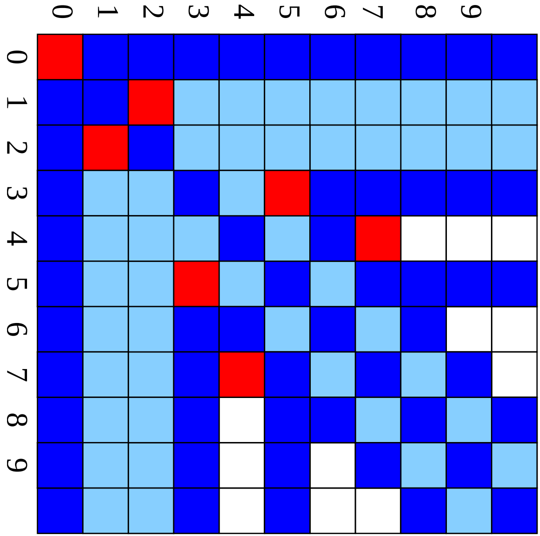
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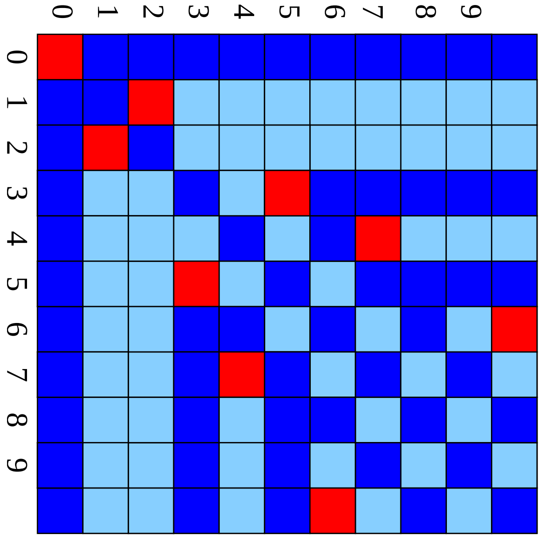
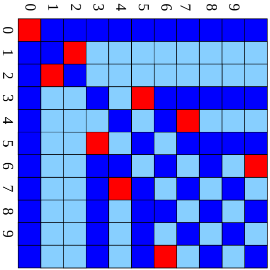

0 1 2 3 4 5 6 7 8 9
0 1 2 3 4 5 6 7 8 9

P-POSITION

A $\mathcal{P}$-position is a position q from which the *previous* player (moving to q) can force a win.



$$(0,0),\ (1,2),\ (3,5),\ (4,7),\ (6,10),\ (8,13),\ (9,15),\ldots$$

W. A. Wythoff, A modification of the game of Nim, *Nieuw Arch. Wisk.* **7** (1907), 199–202.

PROPOSITION

The nth $\mathcal{P}$-position of Wythoff's game is given by

$$(\lfloor n\tau \rfloor, \lfloor n\tau^2 \rfloor) \quad \tau = (1+\sqrt{5})/2.$$

COROLLARY

The $\mathcal{P}$-positions are coded by the Fibonacci word !

$$abaababaabaababaababa\cdots$$

Look at the nth occurrence of a and the nth occurrence of b.

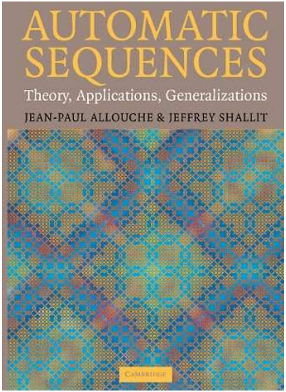
One can also use the Fibonacci numeration system.

Words coding $\mathcal{P}$-positions

- Wythoff's game
  $\leftrightarrow$ Fibonacci word
- A. S. Fraenkel, Flora Game (4 piles)
  $\leftrightarrow$ images by morphisms of the Fibonacci word
  $A \mapsto bac$, $B \mapsto da$.
- E. Duchêne, M.R., A morphic approach to combinatorial games : the Tribonacci case, Theor. Inform. Appl. **42** (2008), 375–393.
  Extension of Wythoff's game on three piles of token
  $\leftrightarrow$ Tribonacci word
- E. Duchêne, M.R., Cubic Pisot Unit Combinatorial Games, Monat. fur Math. **155** (2008), 217 – 249.
  A class of games on three piles
  $\leftrightarrow$ for $s \geq 2$, fixed point of $\sigma : a \mapsto a^s b,\ b \mapsto ac,\ c \mapsto a$.

EXAMPLES OF GAMES RELATED TO MORPHIC WORDS

Let us also mention the well-known game of Nim (with 2 heaps)



p. 448

A two-dimensional array $(a(m,n))_{m,n\geq 0}$ is *k-regular* if there exist a finite number of two-dimensional arrays $(a_i(m,n))_{m,n\geq 0}$ such that each sub-array of the form $(a(k^e m + r, k^e n + s))_{m,n\geq 0}$ with $e \geq 0$, $0 \leq r, s < k^e$ is a $\mathbb{Z}$-linear combination of the a_i.

Sprague-Grundy function : Mex(Opt(p))

	0	1	2	3	4	5	6	7	8	9	⋯
0	0	1	2	3	4	5	6	7	8	9	⋯
1	1	0	3	2	5	4	7	6	9	8	
2	2	3	0	1	6	7	4	5	10	11	
3	3	2	1	0	7	6	5	4	11	10	
4	4	5	6	7	0	1	2	3	12	13	
5	5	4	7	6	1	0	3	2	13	12	
6	6	7	4	5	2	3	0	1	14	15	
7	7	6	5	4	3	2	1	0	15	14	
8	8	9	10	11	12	13	14	15	0	1	
9	9	8	11	10	13	12	15	14	1	0	
⋮	⋮										⋱

Nim-sum: $\sum_{i=0}^{t} a_i\, 2^i \oplus \sum_{i=0}^{t} b_i\, 2^i = \sum_{i=0}^{t} (a_i + b_i \mod 2)\, 2^i$

What about the Sprague-Grundy function for Wythoff's game ?

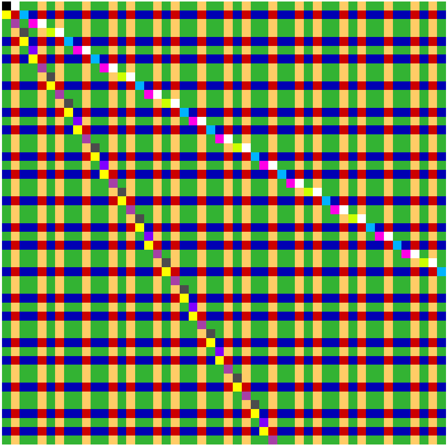
	0	1	2	3	4	5	6	7	8	9	⋯
0	0	1	2	3	4	5	6	7	8	9	⋯
1	1	2	0	4	5	3	7	8	6	10	
2	2	0	1	5	3	4	8	6	7	11	
3	3	4	5	6	2	0	1	9	10	12	
4	4	5	3	2	7	6	9	0	1	8	
5	5	3	4	0	6	8	10	1	2	7	
6	6	7	8	1	9	10	3	4	5	13	
7	7	8	6	9	0	1	4	5	3	14	
8	8	6	7	10	1	1	5	3	4	15	
9	9	10	11	12	8	7	13	14	15	16	
⋮											⋱

OPEN QUESTION

Is there some regular-structure related to this “Wythoff array”?

- A. S. Fraenkel, the Sprague-Grundy function for Wytoff’s game, *Theoret. Comput. Sci.* **75** (1990), 311–333.
- E. Duchêne, A. S. Fraenkel, R. Nowakowski, M.R., Extensions and restrictions of wythoff’s game preserving wythoff’s sequence as set of P-positions, to appear in *J. Combin. Theory Ser. A*.

Examples of games related to morphic words



Informal/arguable definition of what could be
a game that is “easy to play”.

DEFINITION

A possible answer : same options for all positions (provided that enough token are available), i.e., always the same moves. Such a game is called invariant.

REMARK

There is no clear formal framework to decide the quality of given game rules.

Invariant does not necessarily mean “easy”.

EXAMPLES OF INVARIANT GAMES CAN BE FOUND IN...

- game of Nim
- Wythoff's game
- E. Duchêne, S. Gravier, Geometrical extensions of Wythoff's game, *Disc. Math.* **309** (2009), 3595–3608.
- A. S. Fraenkel, Heap games, numeration systems and sequences, *Ann. Comb.* **2** (1998), 197–210.
- A. S. Fraenkel, I. Borosh, A generalization of Wythoff's game, *J. Combin. Theory Ser. A* **15** (1973), 175–191.
- A. S. Fraenkel, D. Zusman, A new heap game, *Theoret. Comput. Sci.* **252** (2001), 5–12.
- Subtraction games given in *Winning ways*

EXAMPLE OF VARIANT GAME : TRIBONACCI GAME

I. Any positive number of tokens from up to two piles can be removed.

II. Let α, β, γ be three positive integers such that

$$2 \max\{\alpha, \beta, \gamma\} \leq \alpha + \beta + \gamma.$$

Then one can remove α (resp. β, γ) from the first (resp. second, third) pile.

III. Let $\beta > 2\alpha > 0$. From position (a, b, c) one can remove the same number α of tokens from any two piles and β tokens from the unchosen one with the following condition. If a' (resp. b', c') denotes the number of tokens in the pile which contained a (resp. b, c) tokens before the move, then the configuration $a' < c' < b'$ is not allowed.

SOME EXAMPLES OF VARIANT GAMES

- A. S. Fraenkel, *The Raleigh game*, Combinatorial number theory, 199–208, de Gruyter, Berlin, (2007).
- A. S. Fraenkel, *The Rat and the Mouse game*.
- A. S. Fraenkel, *The Flora game*.
- *Tribonacci game*, *Cubic Pisot Unit games*, ...

Usually the dependence of the game rules to the actual position is restricted to some simple logical formula.

GENERAL REMARK

Given an infinite sequence $S = (A_n, B_n)$ of nonnegative integers with $(A_0, B_0) = (0, 0)$, a *game on two heaps having S as set of $\mathcal{P}$-positions can always be defined*.

- from any position $(x, y) \notin S$,
  a unique allowed move $(x, y) \rightarrow (0, 0)$
- from any position $(A_n, B_n) \in S$,
  any move is allowed except those leading to another position in S.

QUESTION

Given an infinite sequence $S = (A_n, B_n)$ of nonnegative integers with $(A_0, B_0) = (0, 0)$, find an invariant game on two heaps having S as set of $\mathcal{P}$-positions ?

Our goal : define a family of invariant games.

GET A VARIATION OF WYTHOFF'S GAME BASED ON

- A. S. Fraenkel, How to beat your Wythoff games' opponent on three fronts, *Amer. Math. Monthly* **89** (1982), 353–361.

where is considered an invariant extension of Wythoff's game whose set of $\mathcal{P}$-positions is given by a pair of complementary Beatty sequences based on $\alpha = (1; \overline{k})$.

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad (A_n, B_n) = (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor).$$

- V. Berthé, Autour du système de numération d'Ostrowski, *Bull. Belg. Math. Soc.* **8** (2001), 209)–239.

DEFINITION

The game $G(\alpha_k)$, $k \in \mathbb{N}_{\geq 1}$, has set of moves

$$\mathcal{M}_W \setminus \{(2i,2i) \mid 0 < i < k\} \cup \{(2k+1, 2k+2), (2k+2, 2k+1)\}.$$

Defining an invariant game is easy but we provide "nice" characterizations of the set of $\mathcal{P}$-positions $(A_n, B_n)_{n\geq 0}$

REMARK

For $k = 1$, $\mathcal{M}_W \cup \{(3,4),(4,3)\}$, we have exactly the same $\mathcal{P}$-positions as for Wythoff's game.

EXAMPLE ($k = 2$)

Remove

- any positive number of tokens from one pile or,
- 3 tokens from one pile and 4 from the other one or,
- the same positive number ($\neq 2$) from the two piles.

n	0	1	2	3	4	5	6	7	8	$\cdots$
A_n	0	1	3	5	6	8	10	12	13	$\cdots$
B_n	0	2	4	7	9	11	14	16	18	$\cdots$

Let $k \geq 2$.

RECURSIVE CHARACTERIZATION OF $(A_n, B_n)_{n \geq 0}$

$(A_0, B_0), \ldots, (A_k, B_k) = (0,0), (1,2), (3,4), \ldots, (2k-1, 2k)$,

$$A_n = \mathrm{Mex}\{A_i, B_i \mid i < n\}.$$

For all $n \geq k$, if the following condition holds true

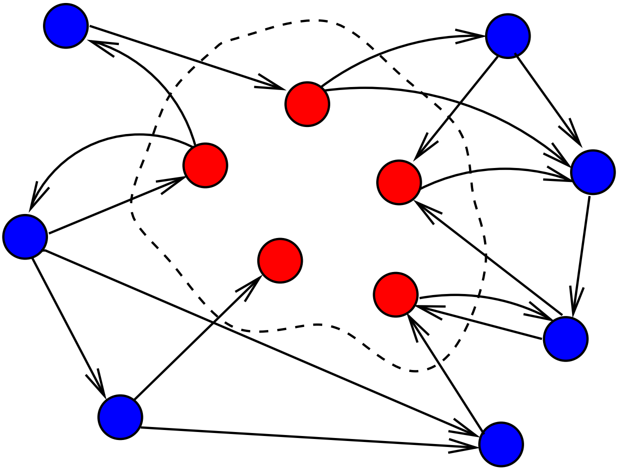
$$A_{n+1} - A_n = 2$$

$$\wedge \quad \Big[(B_n - A_n = B_{n-k+1} - A_{n-k+1} + 1 \wedge A_{n+1} - A_{n-k} \neq 2k+1)$$

$$\vee \; B_{n-k} - A_{n-k} \neq B_n - A_n - 1\Big]$$

then $B_{n+1} - A_{n+1} = B_n - A_n$, otherwise
$B_{n+1} - A_{n+1} = B_n - A_n + 1$.

Proof. The set is stable and absorbing.



An acyclic digraph has a unique kernel.

ALGEBRAIC CHARACTERIZATION OF $(A_n, B_n)_{n\geq 0}$

Let α_k be the quadratic irrational number $(1; \overline{1,k})$ and β_k be such that $\alpha_k^{-1} + \beta_k^{-1} = 1$. Then we have

$$(A_n, B_n) = (\lfloor n\alpha_k \rfloor, \lfloor n\beta_k \rfloor).$$

$$\alpha_k = 1 + \frac{\sqrt{k^2+4k} - k}{2} \in [\frac{1+\sqrt{5}}{2}, 2)$$

and

$$\beta_k = \frac{3}{2} + \frac{\sqrt{k^2+4k}}{2k} \in (2, \frac{3+\sqrt{5}}{2}.$$

Proof. The sequence $(\lfloor n\alpha_k \rfloor, \lfloor n\beta_k \rfloor)$ satisfies the previous recursive definition.

REMARK

We have complementary Beatty sequences.

n	0	1	2	3	4	5	6	7	8	$\cdots$
A_n	0	1	3	5	6	8	10	12	13	$\cdots$
B_n	0	2	4	7	9	11	14	16	18	$\cdots$

$$\Delta_\gamma(n) := \lfloor (n+1)\gamma \rfloor - \lfloor n\gamma \rfloor.$$

The sequence $(\Delta_{\alpha_k}(n))_{n\geq 1}$ (resp. $(\Delta_{\beta_k}(n))_{n\geq 1}$) is a Sturmian sequence over $\{1,2\}$ (resp. $\{2,3\}$).

LEMMAS $\alpha = \alpha_k$, $\beta = \beta_k$

- If $\Delta_\alpha(n) = 1$, then $\Delta_\beta(n) = 2$.
- If $\Delta_\beta(n) = 3$, then $\Delta_\alpha(n) = 2$.
- $\Delta_\alpha(n) \cdots \Delta_\alpha(n+k-1)$ $\in \{2^k\} \cup \{2^i 1 2^{k-i-1} \mid i = 0, \ldots, k-1\}$.
- If $\Delta_\alpha(n) \cdots \Delta_\alpha(n+k-1) = 2^k$, then $\Delta_\beta(n) \cdots \Delta_\beta(n+k-1) \in \{2^i 3 2^{k-i-1} \mid i = 0, \ldots, k-1\}$.
- If $\Delta_\beta(n) \cdots \Delta_\beta(n+k) = 3 2^{k-1} 3$, then $\Delta_\alpha(n) \cdots \Delta_\alpha(n+k) = 2^{k+1}$.

COROLLARY

Thm. 1 (recursive characterization of $(A_n, B_n)_{n\geq 0}$) provides a recursive characterization of a class of Sturmian words :

$$(\Delta_\alpha(n))_{n\geq 0} \text{ and } (\Delta_\beta(n))_{n\geq 0}.$$

CONJECTURE

Given a pair of complementary Beatty sequences $S = (A_n, B_n)$, there exists an invariant game having S as set of P-positions.