A TRIBONACCI GAME

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joint work with Eric Duchêne (post-doc)

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Outline of the talk

What is a combinatorial game ?

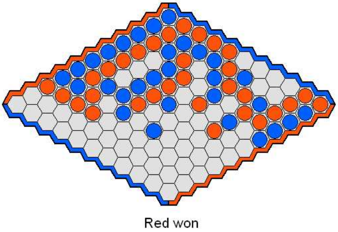
Wythoff's game or "catching the queen"

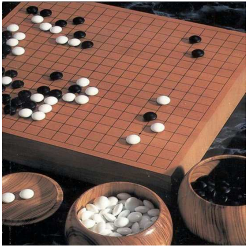
Kernel of the game graph

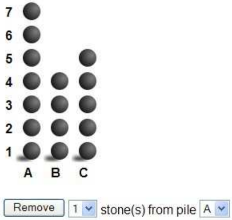
Link with combinatorics on words...

Generalizations of the Wythoff's game

- Reversi, Hex, Go, Game of Nim,. . .

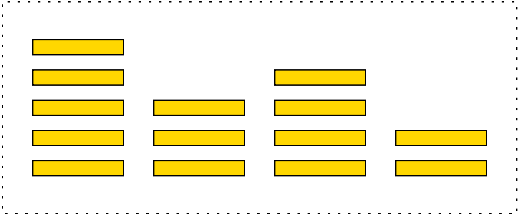






Game of Nim

- Two players
- k piles of tokens, $n_1, \ldots, n_k > 0$
- Players play alternatively and
  remove any number of tokens from *one* pile
- The one who takes the last token wins the game.



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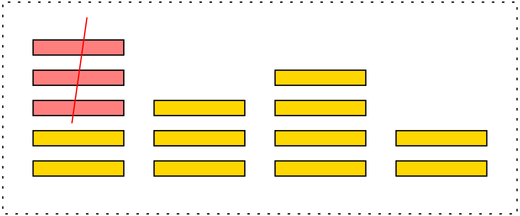

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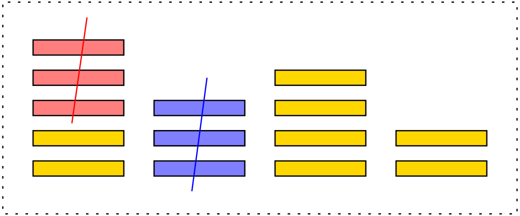

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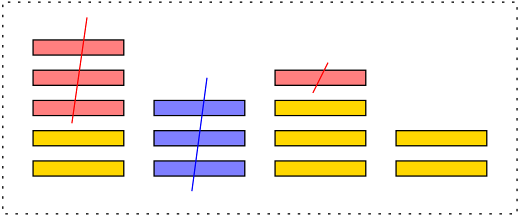

Combinatorial game

- Two players playing alternatively
  (vs. multi-players cooperative games, Nash equilibrium)
- Finite number of configurations
- Well-defined rules (same or different for the 2 players)
- Total information (no hidden card, ...)
- No randomness (no dice, no shuffle, ...)
- "Last move wins"
- Finite game (always ending)

Questions

- Is there a winning strategy for one player ? which one ?
- How can we apply this strategy (complexity issue) ?

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Strategy for the game of Nim, Bouton 1905

A player wins if the addition in base 2 without carry is zero,

$$\bigoplus_{i=1}^{k} n_i = 0$$

Winner is known since the beginning of the game !

We consider the easiest framework:

Impartial (vs. partizan) and acyclic (vs. cyclic) games

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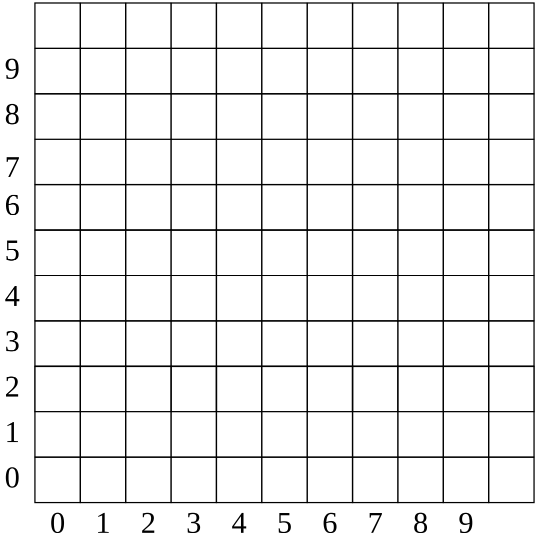
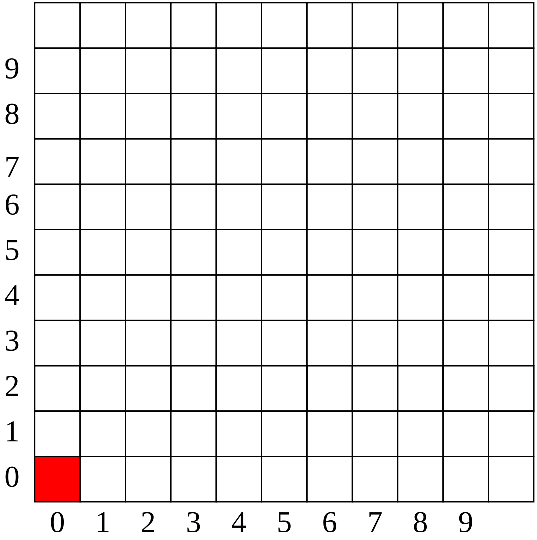
RULES OF THE GAME

- Two players
- 2 piles of tokens
- Players play alternatively
- Remove any number of tokens from one pile or the same number from the two piles
- The one who takes the last token wins the game.

WYTHOFF'S GAME OR "CATCHING THE QUEEN"

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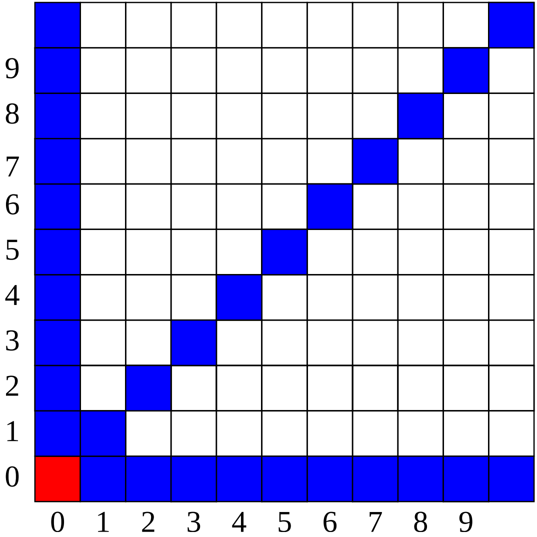

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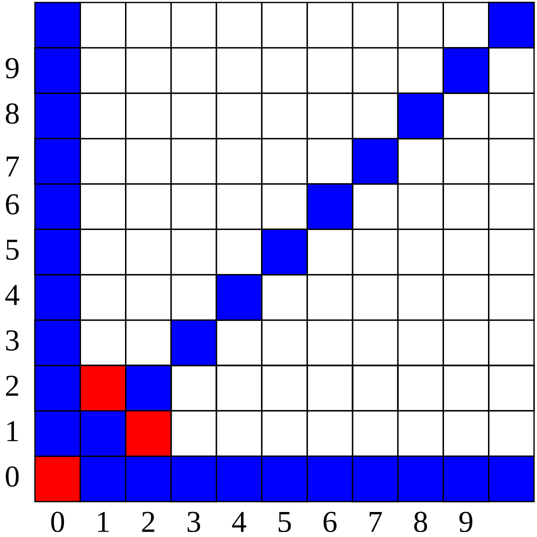

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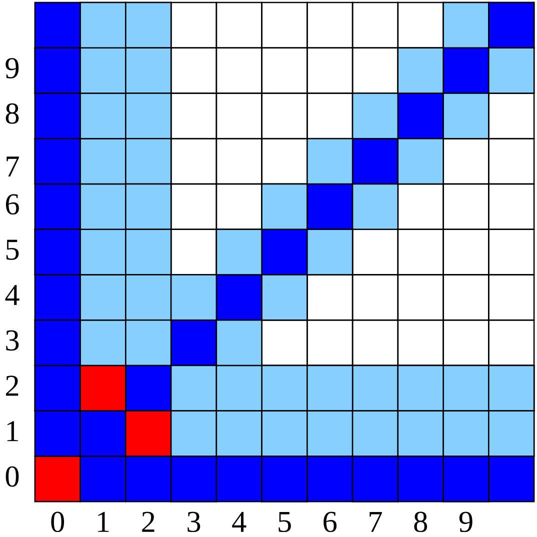

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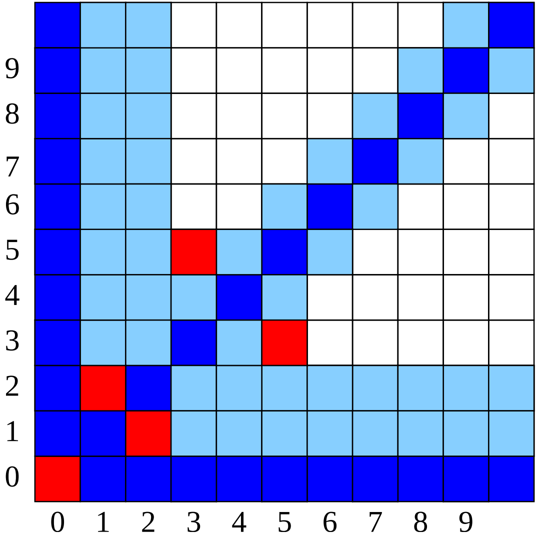

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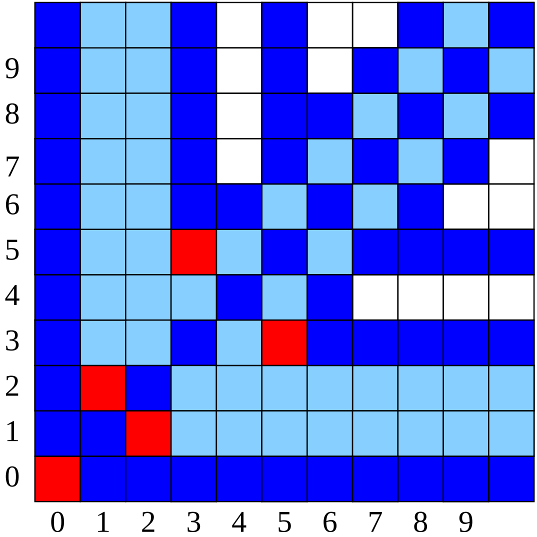

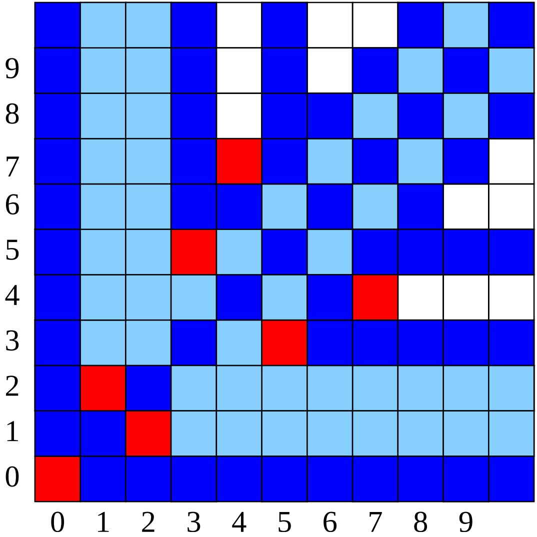
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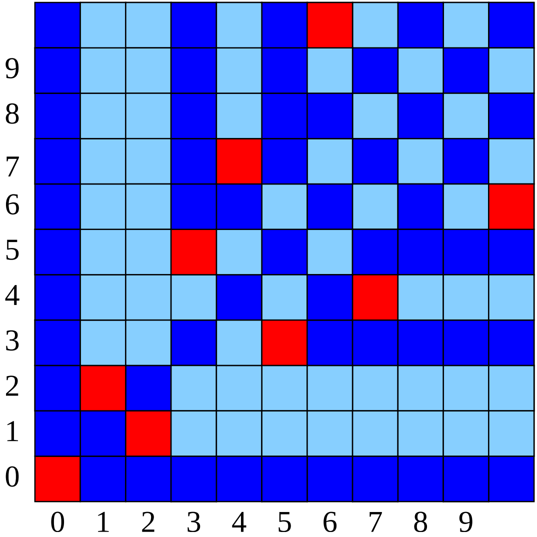
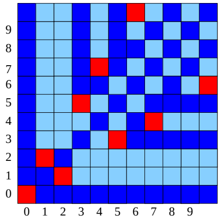




$(0,0),\ (1,2),\ (3,5),\ (4,7),\ (6,10),\ \ldots$

P-POSITION

A P-position is a position q from which the *previous* player (moving to q) can force a win.

PROPERTIES THAT IMPLY WINNING STRATEGY

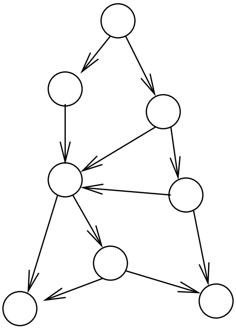
Let $\mathcal{P}$ be the set of P-positions

- $\forall p, q \in \mathcal{P}, p \neq q, p \not\longrightarrow q$ (i.e., stable subset)
- $\forall p \notin \mathcal{P}, \exists q \in \mathcal{P} : p \longrightarrow q$ (i.e., absorbing subset)

TRANSLATION IN GRAPH THEORETICAL TERMS

A kernel is a stable and absorbing subset of vertices. If G is an acyclic (simply connected) graph then G has a unique kernel.

Consider the *game graph* where the vertices represent positions of the game.



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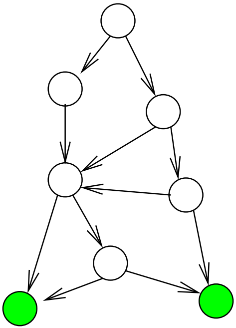

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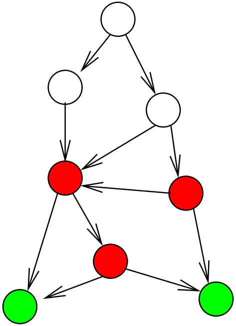

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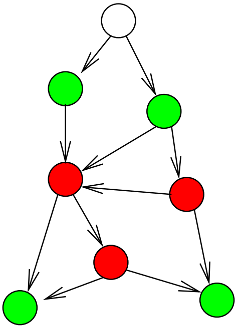

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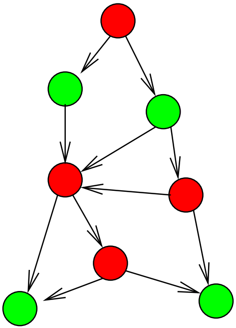

P-POSITION OF THE WYTHOFF'S GAME

$(a_n, b_n)_{n\geq 0} = (0,0),\ (1,2),\ (3,5),\ (4,7),\ \ldots$

$$(*) \quad \forall n \geq 0, \quad \begin{cases} a_n = Mex\{a_i, b_i \mid i < n\} \\ b_n = a_n + n \end{cases}$$

Mex stands for "Minimum EXcluded value", e.g., $Mex(\emptyset) = 0$ and $Mex(\{0,1,2,4,7\}) = 3$

QUESTION ?

Let a_n (resp. b_n) denote the position of the n-th letter a (resp. b) in an infinite word over $\{a, b\}$. Do you know an infinite word satisfying (*) for all $n \geq 1$?

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P-POSITIONS OF THE WYTHOFF'S GAME

$$(a_n, b_n)_{n\geq 0} = (\lfloor n\,\tau\rfloor, \lfloor n\,\tau^2\rfloor).$$

A. FRAENKEL, TO APPEAR IN INTEGERS

The Raleigh Game, a game on three piles...

A. FRAENKEL, ANNALS OF COMBIN.'98

Heap games, numeration systems and sequences.

QUESTION ?

Is there a game (on three piles) whose set of P-positions is given by the Tribonacci word ?

$$\mathbf{t} = abacabaabacababacabaabacabacabaabacabab\cdots.$$

DUCHÊNE, M.R.

I. Any positive number of tokens from up to two piles can be removed.

II. Let $\alpha, \beta, \gamma > 0$ such that $2\max\{\alpha, \beta, \gamma\} \leq \alpha + \beta + \gamma$. Then one can remove α (resp. β, γ) from the first (resp. second, third) pile.

III. Let $\beta > 2\alpha > 0$. From (a, b, c) one can remove the same number α of tokens from any two piles and β tokens from the unchosen one. But the configuration $a' < c' < b'$ is not allowed.

wait until slide 37…

$\mathbf{t} = abacabaabacababacabaabacabacabaabacabab \cdots .$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{A_n}$	0	1	3	5	7	8	10	12	14	16	18	20	21
$\mathbf{B_n}$	0	2	6	9	13	15	19	22	26	30	33	37	39
$\mathbf{C_n}$	0	4	11	17	24	28	35	41	48	55	61	68	72

This sequence was studied in

- L. Carlitz, R. Scoville, V.E. Hoggatt Jr., Fibonacci representations of higher order, *Fibonacci Quart.* **10** (1972), 43–69.
- E. Barcucci, L. Bélanger, S. Brlek, On Tribonacci sequences, *Fibonacci Quart.* **42** (2004), 314–319.

$\Delta_n(a) := A_{n+1} - A_n = \psi_a(t_n)$,
$\Delta_n(b) := B_{n+1} - B_n = \psi_b(t_n)$ and
$\Delta_n(c) := C_{n+1} - C_n = \psi_c(t_n)$
where

	ψ_a	ψ_b	ψ_c
a	2	4	7
b	2	3	6
c	1	2	4

$$\Delta_n(c) = \Delta_n(a) + \Delta_n(b) + 1$$

because

	τ	τ^2	τ^3
a	ab	$abac$	$abacaba$
b	ac	aba	$abacab$
c	a	ab	$abac$

e.g., $\tau^2(b\,*) = aba\,\underline{a}\,\underline{ab}\cdots$

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a	ab	$abac$	$abacaba$	
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c	a	ab	$abac$	

CHARACTERIZATION OF THE TRIBONACCI WORD

Assume X_n denotes the position of the n-th letter x occurring in the Tribonacci word **t**, $X = A, B, C$ (resp. $x = a, b, c$).

$$\forall n \geq 1, \quad \begin{cases} A_n &= Mex\{A_i, B_i, C_i : 0 \leq i < n\} \\ B_n &= A_n + Mex\{B_i - A_i, C_i - B_i : 0 \leq i < n\} \\ C_n &= A_n + B_n + n. \end{cases}$$

ARGUMENT

$$C_n = C_1 + \sum_{i=1}^{n-1} \Delta_i(c) = C_1 + \sum_{i=1}^{n-1} (\Delta_i(a) + \Delta_i(b) + 1)$$

same argument for the Fibonacci / n-bonacci words.

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Proposition

Assume X_n denotes the position of the n-th letter x occurring in the Tribonacci word **t**, $X = A, B, C$ (resp. $x = a, b, c$). The sets

$$\{B_i - A_i \mid i \geq 1\} = \{1, 3, 4, 6, 7, 9, 10, 12, 14, 15, 17, 18, 20, \ldots\}$$

$$\{C_i - B_i \mid i \geq 1\} = \{2, 5, 8, 11, 13, 16, 19, 22, 25, 28, 31, \ldots\}$$

give a partition of $\mathbb{N}_{\geq 1}$.

A word about the proof. . .

Definition

Let $\mathbf{x} = (x_n)_{n\geq 0} \in \mathbb{N}^{\mathbb{N}}$, $S(\mathbf{x})$ is the corresponding summatory sequence, i.e., for all $n \geq 0$,

$$S(\mathbf{x})_n = \sum_{i=0}^{n} x_i.$$

Proposition

Let $\mu : \{a, b, c\} \to \{2, 3\}$ and $\nu : \{a, b, c\} \to \{1, 2\}^*$ s.t.

$$\mu(a) = \mu(b) = 3,\ \mu(c) = 2,\ \nu(a) = \nu(b) = 21 \text{ and } \nu(c) = 2.$$

For any infinite word $w \in \{a, b, c\}^\omega$, the sets

$$S_\mu = \{S(2\,\mu(w))_n \mid n \geq 0\} \quad \text{and} \quad S_\nu = \{S(1\,\nu(w))_n \mid n \geq 0\}$$

form a partition of $\mathbb{N}_{\geq 1}$.

$\mu(a) = \mu(b) = 3,\ \mu(c) = 2,\ \nu(a) = \nu(b) = 21$ and $\nu(c) = 2$.

	a	b	a	c	b	b	a	c
$\chi_\mu =$ 01	001	001	001	01	001	001	001	01
$\chi_\nu =$ 1	011	011	011	01	011	011	011	01

01001001001010010010010101
1011011011010110110110101

Corollary

- $C_1 - B_1 = 2$,
- $(C_{n+1} - B_{n+1}) - (C_n - B_n) = \Delta_n(c) - \Delta_n(b)$
  $= \psi_c(t_n) - \psi_b(t_n) = \mu(t_n)$

So $\{C_i - B_i \mid i \geq 1\} = \{S(2\,\mu(\mathbf{t}))_n \mid n \geq 0\}$,
and proceed in the same way for $\{B_i - A_i \mid i \geq 1\}\ldots$

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$\chi_\mu = $ 01	001	001	001	01	001	001	001	01
$\chi_\nu = $ 1	011	011	011	01	011	011	011	01

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COROLLARY

- $C_1 - B_1 = 2,$
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So $\{C_i - B_i \mid i \geq 1\} = \{S(2\,\mu(\mathbf{t}))_n \mid n \geq 0\}$,
and proceed in the same way for $\{B_i - A_i \mid i \geq 1\}\ldots$

Remember our game on three piles (see slide 30)

THEOREM

The set $\mathcal{S} = \{(A_n, B_n, C_n) \mid n \geq 0\}$ is the set of P-positions of the Tribonacci game.

IDEA OF THE PROOF

- Whatever the rule is, a player moving $(A_n, B_n, C_n) \in \mathcal{S}$ always lands in a position not in $\mathcal{S}$.
- Given a position $(a, b, c) \notin \mathcal{S}$, there exists a move to some $(A_n, B_n, C_n) \in \mathcal{S}$.

In graph theoretical terms, we show that the set of P-positions is the kernel of the acyclic game graph.

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We need some technical results like (see Carlitz *et al.*)

$$\forall n \geq 1, \quad C(n) - B(n) = 1 + B(A(n)) - A(A(n))$$

$$\forall n \geq 0, \quad C(n) - B(n) + 1 = B(A(n)+1) - A(A(n)+1),$$

$$\forall n \geq 0, \quad B(n) = A(A(n)+1) - 1.$$

QUESTION

Does (a, b, c) belong to $\mathcal{S} = \{(A_n, B_n, C_n) \mid n \geq 0\}$?

NAIVE ALGORITHM

Produce all the (A_n, B_n, C_n)'s up to a.

Size of the input : $\log_k a$.

This algorithm is polynomial in a...
but exponential with respect to $\log_k a$!

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Tribonacci numeration

$T_0 = 1$, $T_1 = 2$, $T_3 = 4$ and $T_{k+3} = T_{k+2} + T_{k+1} + T_k$

$$n = \sum_{i=0}^{\ell} c_i\, T_i, \quad \rho_T(n) = c_\ell \cdots c_0$$

with $c_i \in \{0, 1\}$, $c_\ell = 1$ and no 111's.

A_n	B_n	C_n	$\rho_T(A_n - 1)$	$\rho_T(B_n - 1)$	$\rho_T(C_n - 1)$
1	2	4	ε	1	11
3	6	11	10	101	1011
5	9	17	100	1001	10011
7	13	24	110	1101	11011

Theorem

(a, b, c) is a P-position of the Tribonacci game iff there exists a word $w \in \{0, 1\}^*$ such that w does not contain 111 and

$$\rho_T(a-1) = w0, \quad \rho_T(b-1) = w01 \quad \text{and} \quad \rho_T(c-1) = w011.$$