Extensions and restrictions of Wythoff’s game preserving Wythoff’s sequence as set of $P$ positions

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Wythoff’s game or “catching the queen”


**Rules of the game**

- Two players play alternatively
- Two piles of tokens
- Remove
  - any positive number of tokens from one pile or,
  - the same positive number from the two piles.
- The one who takes the last token wins the game (**last move wins**).

Set of moves: \( \{(i, 0), \ i > 0\} \cup \{(0, j), \ j > 0\} \cup \{(k, k), \ k > 0\} \)
Wythoff’s game or “catching the queen”
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Wythoff’s game or “catching the queen”
Wythoff’s Game or “Catching the Queen”

\[(0, 0), (1, 2), (3, 5), (4, 7), (6, 10), \ldots\]

**P-position**

A **P-position** is a position \( q \) from which the previous player (moving to \( q \)) can force a win.

**N-position**

A **N-position** is a position \( p \) from which the actual player has an option leading ultimately to win the game.

**Question:** Are all positions \( N \) or \( P \)?
**GAME GRAPH**

Initial position \((i_0, j_0)\), by symmetry, take only \((i \geq j)\)

- **Vertices**: \(\{(i, j), \ i \leq i_0, \ j \leq j_0\}\)
- **Edges**: from each position to all its options:

\[
\begin{align*}
  i > 0 & \quad (i, j) \to (i-k, j) \quad k = 1, \ldots, i \\
  j > 0 & \quad (i, j) \to (i, j-k) \quad k = 1, \ldots, j \\
  i, j > 0 & \quad (i, j) \to (i-k, j-k) \quad k = 1, \ldots, \min(i, j)
\end{align*}
\]
**Remark**

Due to the rules, the game graph for Wythoff’s game is acyclic.

**Theorem [Berge]**

Any finite acyclic digraph has a unique kernel. Moreover, this kernel can be obtained efficiently.

**Reminder/Definition of a Kernel**

A kernel in a graph $G = (V, E)$ is a subset $W \subseteq V$

- **stable**: $\forall x, y \in W$, $(x, y) \notin E$
- **absorbing**: $\forall x \in V \setminus W$, $\exists y \in W : (x, y) \in E$. 
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Bottom-Up approach from the sinks
(they belong to the kernel because it is absorbing)
GAME GRAPH

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Bottom-Up approach from the sinks
(they belong to the kernel because it is absorbing)
For Wythoff’s game, its game graph has a unique kernel $K$.

- **stable**: from a position in $K$, you always play out of $K$,
- **absorbing**: from a position outside $K$, you can play into $K$,
- $(0, 0)$ has to belong to $K$, otherwise $K$ won’t be absorbing.

**Corollary**

The set of $\mathcal{P}$-positions is exactly the kernel $K$ and all the other positions are $\mathcal{N}$-positions.

$$\{\mathcal{P}\text{-positions}\} \supseteq K$$

If $p$ is a position in $K$, then it is a $\mathcal{P}$-position because there is a *winning strategy* outside $K$.

$$\{\mathcal{P}\text{-positions}\} \subseteq K$$

If $p$ is a $\mathcal{P}$-position not in $K$, then there is a move from $p$ to $K$, thus $p$ is a $\mathcal{N}$-position!
**GAME GRAPH**

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**COROLLARY**

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**P-position of the Wythoff’s game I**

\((A_n, B_n)_{n \geq 0} = (0, 0), (1, 2), (3, 5), (4, 7), \ldots\)

\[
\forall n \geq 0, \quad \begin{cases} 
A_n = \text{Mex}\{A_i, B_i \mid i < n\} \\
B_n = A_n + n 
\end{cases}
\]

**P-position of the Wythoff’s game II**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | \ldots |
|---|---|---|---|---|---|---|---|---|-----|-----|-----|-----|-----|
| F | a | b | a | a | b | a | b | a | a | b | a | a | b | a |

**P-positions of the Wythoff’s game III**

\((A_n, B_n)_{n \geq 0} = ([n \tau], [n \tau^2]).\)
Many variations of the Wythoff’s game


Different sets of moves / more piles

\[ \downarrow \]

Different sets of \( P \)-positions to characterize...
Consider extensions or restrictions of Wythoff’s game that keep the set of \( P \)-positions of Wythoff’s game invariant.

Characterize the different sets of moves...

\[ \downarrow \]

Same set of \( P \)-positions as Wythoff’s game
Canonical construction [Cobham’72]: morphisms $\rightarrow$ automata

$$\varphi : a \mapsto abc, \ b \mapsto ac, \ c \mapsto b$$

Consider the language $L = L\left(\mathcal{M}\right) \setminus \{0, 1, 2\}^*$. 

Remark: Positions in $\varphi^\omega(a)$ are counted from 1.
Take the words of $L$ in genealogical order (abstract system)

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Not a “positional” system, no sequence behind.

**Example:**

The 4th letter is $a$, it corresponds to $w_3 = 10$.

Since $\varphi(a) = abc$, we consider $w_3 0 = 100 = w_i$

$w_3 1 = 101 = w_{i+1}$

$w_3 2 = 102 = w_{i+2}$

then the $(i + 1)$st, $(i + 2)$st, $(i + 3)$st letters are $a, b, c$. 
\[ \text{rep}_L(i) := w_i, \quad \text{val}_L(w_i) := i \]

**Proposition**

Let the \( n \)th letter of \( \varphi^\omega(a) \) be \( \sigma \) and \( w_{n-1} \) be the \( n \)th word in \( L \). If \( \varphi(\sigma) = x_1 \cdots x_r \), then \( x_1 \cdots x_r \) appears in \( \varphi^\omega(a) \) in positions \( \text{val}_L(w_{n-1}x_1) + 1, \ldots, \text{val}_L(w_{n-1}x_r) + 1 \).

For Wythoff’s game: Fibonacci word \( \mathcal{F} \), \( L = 1\{01, 0\}^* \cup \{\varepsilon\} \) and we get the usual Fibonacci system \( \rho_\mathcal{F} : \mathbb{N} \to L \), \( \pi_\mathcal{F} : L \to \mathbb{N} \).

**Corollary**

- If the \( n \)th letter in \( \mathcal{F} \) is \( a \) (\( n \geq 1 \)), then this \( a \) produces through \( \varphi \) a factor \( ab \) occupying positions \( \pi_\mathcal{F}(\rho_\mathcal{F}(n-1)0) + 1 \) and \( \pi_\mathcal{F}(\rho_\mathcal{F}(n-1)1) + 1 \).

- If the \( n \)th letter in \( \mathcal{F} \) is \( b \) (\( n \geq 1 \)), then this \( b \) produces through \( \varphi \) a letter \( a \) occupying position \( \pi_\mathcal{F}(\rho_\mathcal{F}(n - 1)0) + 1 \).
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- If the \( n \)th letter in \( \mathcal{F} \) is \( a \) \( (n \geq 1) \), then this \( a \) produces through \( \varphi \) a factor \( ab \) occupying positions \( \pi_F(\rho_F(n-1)0)+1 \) and \( \pi_F(\rho_F(n-1)1)+1 \).
- If the \( n \)th letter in \( \mathcal{F} \) is \( b \) \( (n \geq 1) \), then this \( b \) produces through \( \varphi \) a letter \( a \) occupying position \( \pi_F(\rho_F(n-1)0)+1 \).
Reminder on Fibonacci numeration system

Fibonacci sequence: \( F_{i+2} = F_{i+1} + F_i, \ F_0 = 1, \ F_1 = 2 \)

Use greedy expansion, \( \ldots, 21, 13, 8, 5, 3, 2, 1 \)

\[
\begin{array}{c|c|c|c|c}
  n & \rho_F(n) & n & \rho_F(n) & n & \rho_F(n) \\
  1 & 1 & 8 & 10000 & 15 & 100010 \\
  2 & 10 & 9 & 10001 & 16 & 100100 \\
  3 & 100 & 10 & 10010 & 17 & 100101 \\
  4 & 101 & 11 & 10100 & 18 & 101000 \\
  5 & 1000 & 12 & 10101 & 19 & 101001 \\
  6 & 1001 & 13 & 100000 & 20 & 101010 \\
  7 & 1010 & 14 & 100001 & 21 & 1000000 \\
\end{array}
\]

In fact, this is a special case of the following result.

**Theorem [A. Maes, M.R. ’02]**

The set of $S$-automatic sequences is exactly the set of morphic words.

Take any regular language genealogically ordered $\oplus$ DFAO

$$
\begin{array}{c|cccccccccc}
   i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\
\hline
   \text{rep}_S(i) & \varepsilon & a & b & aa & ab & bb & aaa & aab & abb & bbb & \cdots \\
\end{array}
$$

01023031200231010123023031203120231002310123010123\cdots
For all $n \geq 1$, we have

$$A_n = \pi_F(\rho_F(n - 1)0) + 1$$

$$B_n = \pi_F(\rho_F(A_n - 1)1) + 1.$$
More?

Can we get a “morphic characterization” of the Wythoff’s matrix?

\[(P_{i,j})_{i,j \geq 0} = \]

\[
\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\
\vdots & & & & & & & & & & & \vdots
\end{array}
\]
Let’s try something...

$$\varphi : a \mapsto \begin{array}{cc} a & b \\ c & d \end{array}, \quad b \mapsto \begin{array}{c} i \\ e \end{array}, \quad c \mapsto \begin{array}{cc} i & j \end{array}, \quad d \mapsto \begin{array}{c} i \\ e \mapsto \begin{array}{cc} f & b \end{array} \end{array}$$

$$f \mapsto \begin{array}{cc} g & b \\ h & d \end{array}, \quad g \mapsto \begin{array}{cc} f & b \\ h & d \end{array}, \quad h \mapsto \begin{array}{cc} i & m \end{array}, \quad i \mapsto \begin{array}{cc} i & m \\ h & d \end{array}$$

$$j \mapsto \begin{array}{c} k \\ c \end{array}, \quad k \mapsto \begin{array}{cc} l & m \end{array}, \quad l \mapsto \begin{array}{cc} k & m \end{array}, \quad m \mapsto \begin{array}{c} i \\ h \end{array}$$

and the coding

$$\mu : e, g, j, l \mapsto 1, \quad a, b, c, d, f, h, i, k, m \mapsto 0$$
SHAPE-SYMMETRIC MORPHISM [A. MAES ’99]

If \( P \) is the infinite bidimensional picture that is the fixpoint of \( \varphi \), then for all \( i, j \in \mathbb{N} \), if \( \varphi(P_{i,j}) \) is a block of size \( k \times \ell \) then \( \varphi(P_{j,i}) \) is of size \( \ell \times k \)
sizes : 1, 2, 3, 5
```
\begin{array}{cccccc}
  a & b & i & i & m & i \\
  c & d & e & h & d & h \\
  i & j & i & f & b & i \\
  i & m & k & i & m & g \\
  i & m & i & l & m & i \\
  h & d & h & c & d & h \\
  i & m & i & i & j & i \\
\end{array}
```

size : 8, ...
We can do the same as for the unidimensional case: Automaton with input alphabet

\[ \{ (0,0), (1,0), (0,1), (1,1) \} \]

\[ \varphi(r) = \begin{array}{c|c}
    s & t \\
    u & v 
  \end{array}, \quad \begin{array}{c|c}
    s & t \\
    u & \ 
  \end{array}, \quad \begin{array}{c|c}
    s & \\
    u & \ 
  \end{array} \quad \text{or} \quad \begin{array}{c|c}
    s & \\
    & \ 
  \end{array} \]

we have transitions like

\[ r \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow s, \quad r \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow t, \quad r \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow u, \quad r \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow v. \]
We get (after trimming useless part with four states)

This automaton accepts the words

\[
\begin{pmatrix}
0w_1 \cdots w_\ell \\
w_1 \cdots w_\ell 0
\end{pmatrix}
\] and

\[
\begin{pmatrix}
w_1 \cdots w_\ell 0 \\
0w_1 \cdots w_\ell
\end{pmatrix}
\]

where \(w_1 \cdots w_\ell\) is a valid \(F\)-representation ending with an even number of zeroes.
Such a characterization is well-known, but differs from the one we get previously...

**Reminder**

For all $n \geq 1$, we have

\[
A_n = \pi_F(\rho_F(n-1)0) + 1 \\
B_n = \pi_F(\rho_F(A_n - 1)1) + 1.
\]

It is hopefully the same, but **why** ?
• First case: \( \rho_F(n - 1) = u0 \)

\[
\rho_F(A_n) = \rho_F(\pi_F(\rho_F(n - 1)0) + 1) = u01 \text{ no zero}
\]

\( \rho_F(A_n - 1) = u00 \) and

\[
\rho_F(B_n) = \rho_F(\pi_F(\rho_F(A_n - 1)1) + 1) = u010 \text{ one zero}
\]

• Second case: \( \rho_F(n - 1) = u01 \)

\[
\rho_F(A_n) = \rho_F(\pi_F(\rho_F(n - 1)0) + 1) = "u011" \ldots
\]

Normalize \( u011 \) or look for the successor of \( u010 \)
Use the transducer (R to L) computing the successor [Frougny’97]

\[ 10 \rightarrow 100, \quad 2 \text{ zeroes} \]

\[ x10(01)^n010 \rightarrow x101(00)^n00 \quad 2n + 2 \text{ zeroes}, \quad n \geq 0 \]

\[ 1(01)^n010 \rightarrow 100(00)^n00 \quad 2n + 4 \text{ zeroes}, \quad n \geq 0 \]
\( \rho_F(A_n - 1) = u010 \) and

\[
\rho_F(B_n) = \rho_F(\pi_F(\rho_F(A_n - 1)1) + 1) = "u0102" \ldots
\]

101 → 1000, 3 zeroes

\[
x10(01)^n 0101 \rightarrow x101(00)^n000 \quad 2n + 3 \text{ zeroes, } n \geq 0
\]

\[
1(01)^n 0101 \rightarrow 100(00)^n000 \quad 2n + 5 \text{ zeroes, } n \geq 0
\]

Conclusion: “\( A_n \) even number of zeroes, \( B_n \) one more”, OK
EXTENSION PRESERVING SET OF $\mathcal{P}$-POSITIONS

To decide whether or not a move can be adjoined to Wythoff’s game without changing the set $K$ of $\mathcal{P}$-positions, it suffices to check that it does not change the stability property $K$.

Remark: absorbing property holds true whatever the adjoined move is.

CONSEQUENCE

A move $(i, j)$ can be added IFF it prevents to move from a $\mathcal{P}$-position to another $\mathcal{P}$-position.

In other words, a necessary and sufficient condition for a move $(i, j)_{i<j}$ to be adjoined is that it does not belong to

$\{(A_n-A_m, B_n-B_m) : n > m \geq 0\} \cup \{(A_n-B_m, B_n-A_m) : n > m \geq 0\}$
Thanks to the previous characterizations of $A_n$, $B_m$,

**Proposition**

A move $(i, j)_{i<j}$ can be adjoined to without changing the set of $P$-positions IFF

$$(i, j) \neq ([n\tau] - [m\tau], [n\tau^2] - [m\tau^2]) \quad \forall n > m \geq 0$$

and

$$(i, j) \neq ([n\tau] - [m\tau^2], [n\tau^2] - [m\tau]) \quad \forall n > m \geq 0$$
For all $i, j \geq 0$, $W_{i,j} = 1$ IFF Wythoff’s game with the adjoined move $(i, j)$ has Wythoff’s sequence as set of $\mathcal{P}$-positions,

\[
(W_{i,j})_{i,j \geq 0} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]
COROLLARY

Let $I \subseteq \mathbb{N}$. Wythoff’s game with adjoined moves

$$\{(x_i, y_i) : i \in I, x_i, y_i \in \mathbb{N}\}$$

has the same sequence $(A_n, B_n)$ as set of $\mathcal{P}$-positions

IFF

$W_{x_i,y_i} \neq 1$ for all $i \in I$. 
Complexity issue

We investigate tractable extensions of Wythoff’s game, we also need to test these conditions in polynomial time. And the winner can consummate a win in at most an exponential number of moves.

Many “efforts” lead to this

For any pair $(i, j)$ of positive integers, we have $W_{i,j} = 1$ if and only if one the three following properties is satisfied:

1. $(\rho_F(i - 1), \rho_F(j - 1)) = (u0, u01)$ for any valid $F$-representation $u$ in $\{0, 1\}^*$.
2. $(\rho_F(i - 2), \rho_F(j - 2)) = (u0, u01)$ for any valid $F$-representation $u$ in $\{0, 1\}^*$.
3. $(\rho_F(j - A_i - 2), \rho_F(j - A_i - 2 + i)) = (u1, u'0)$ for any two valid $F$-representations $u$ and $u'$ in $\{0, 1\}^*$. 
Morphic characterization of $\mathcal{W}$... in progress

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and the coding \( \nu : a, b, c, d, e, i, j, k, l, n, o, p, q, r \mapsto 0 \)
\( f, g, h, m, s, t, u, v, w, x, y, z \mapsto 1. \)
Corresponding automaton
SOME OF THE MACHINERY BEHIND
**Lemma**

Let $F_n$ be the prefix of $F$ of length $n$. For any finite factor $bua$ occurring in $F$ with $|u| = n$, we have $|u|_a = |F_n|_a$ and $|u|_b = |F_n|_b$.

**Example**

Take $u = aabaab$, $bua$ of length 8 starts in $F$ from position 7. $F_6 = abaaba$ is a permutation of $u$.

\[ F = \begin{array}{c}
abaaba b aabaab a baababaaba \cdots \\
F_6 & u
\end{array} \]

Proof: algebraic
**Lemma**

Let $n \geq 1$ be such that $B_{n+1} - B_n = 2$. Then $\rho_F(B_n - 1)$ ends with 101.

Proof: Morphic structure of $\mathcal{F}$

**Proposition**

$$\{(A_j - A_i, B_j - B_i) \mid j > i \geq 0\} = \{(A_n, B_n) \mid n > 0\}$$

$$\cup \{(A_n + 1, B_n + 1) \mid n > 0\}$$

Proof: Density of the $\{n\tau\}$'s in $[0, 1]$
**Lemma**

Let $u_1 \in \{0, 1\}^*$ be a valid $F$-representation. If $\rho_F(\pi_F(u_1) + n)1$ is also a valid $F$-representation, then

$$\pi_F(\rho_F(\pi_F(u_1) + n)1) = \pi_F(u00) + \pi_F(\rho_F(n - 1)0) + 4.$$  

Otherwise, $\rho_F(\pi_F(u_1) + n)1$ is not a valid $F$-representation and

$$\pi_F(\rho_F(\pi_F(u_1) + n)0) = \pi_F(u00) + \pi_F(\rho_F(n)0) + 2.$$  

**Proof:** Morphic structure of $\mathcal{F}$

**Theorem**

Let $i, j$ be such that $A_j - B_i = n > 0$. We have

$$B_j - A_i = B_i + A_n + 1.$$
CONCLUDING RESULT

THEOREM

There is no redundant move in Wythoff’s game. In particular, if any move is removed, then the set of $P$-positions changes.