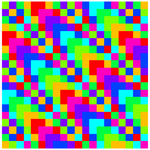
State complexity of testing divisibility...

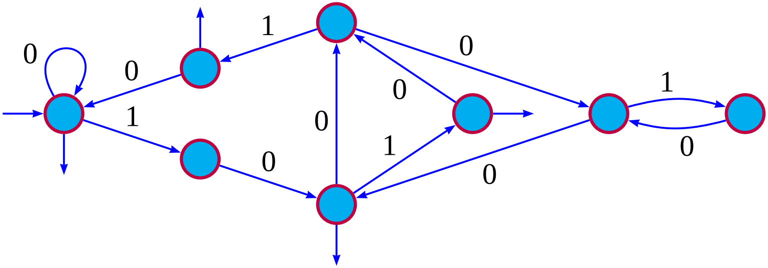
É. Charlier, N. Rampersad, M. Rigo, L. Waxweiler
(University of Liège)

http://www.discmath.ulg.ac.be/

Numeration 2010 – Leiden, 18th June 2010



WHAT IS THIS TALK ABOUT ?



13	8	5	3	2	1	
				1	0	2
			1	0	1	4
		1	0	0	1	6
	1	0	0	0	0	8
	1	0	0	1	0	10
	1	0	1	0	1	12
						⋮

What is this talk about ?

The set $2\mathbb{N}$ of even integers is *F-recognizable* or *F-automatic*, i.e., the language $\mathrm{rep}_F(2\mathbb{N}) = \{10, 101, 1001, 10000, \ldots\}$ is accepted by some finite automaton.

Remark (in terms of Chomsky's hierarchy)

With respect to the Fibonacci system, *any* F-recognizable set can be considered as a "*particularly simple*" set of integers.

We get a similar definition for other numeration systems.

A NUMERATION SYSTEM

- A *numeration system* is an increasing sequence of integers $U = (U_n)_{n\geq 0}$ such that
  - $U_0 = 1$ and
  - $C_U := \sup_{n\geq 0} \lceil U_{n+1}/U_n \rceil < \infty$.
- U is linear if it satisfies a linear recurrence relation over $\mathbb{Z}$, for all large enough n.

EXAMPLE

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence with $F_0 = 1$ and $F_1 = 2$.

Let $A \subset \mathbb{Z}$ be an alphabet.
For any word $w = w_{\ell-1} \cdots w_0$ over A, we set

$$\mathrm{val}_{A,U}(w) := \sum_{i=0}^{\ell-1} w_i \, U_i .$$

Greedy representations

- A greedy representation of a non-negative integer n is a word $w = w_{\ell-1} \cdots w_0$ over $A_U = \{0, 1, \ldots, C_U - 1\}$ such that

$$\mathrm{val}_U(w) = \sum_{i=0}^{\ell-1} w_i\, U_i = n,$$

and for all j

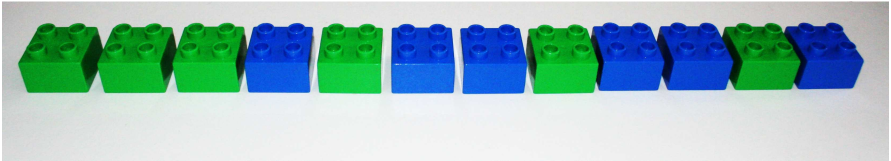
$$\sum_{i=0}^{j-1} w_i\, U_i < U_j.$$

- $\mathrm{rep}_U(n)$ is the greedy representation of n with $w_{\ell-1} \neq 0$.
- $X \subseteq \mathbb{N}$ is U-recognizable, if $\mathrm{rep}_U(X)$ is accepted by a finite automaton.

Motivations

- Cobham's theorem for integer base systems (1969) shows that *recognizability depends on the choice of the base*. Only ultimately periodic sets are recognizable in all bases.
- Introduction of non-standard numeration systems and study U-recognizable sets.
- Ultimately periodic sets are still U-recognizable for any numeration system U such that $\mathbb{N}$ is U-recognizable.

- V. Bruyère, G. Hansel, C. Michaux, R. Villemaire, Logic and p-recognizable sets of integers, *BBMS* **1** (1994).
- V. Bruyère, G. Hansel, Bertrand numeration systems and recognizability, *TCS* **181** (1997).



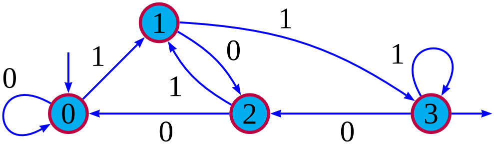
Motivations

Proposition

Let $p, r \geq 0$. If $(U_n)_{n\geq 0}$ is a linear numeration system, then

$$\mathrm{val}^{-1}_{A_U,U}(p\,\mathbb{N}+r) = \left\{ c_\ell \cdots c_0 \in A_U^* \mid \sum_{k=0}^{\ell} c_k\, U_k \in p\,\mathbb{N}+r \right\}$$

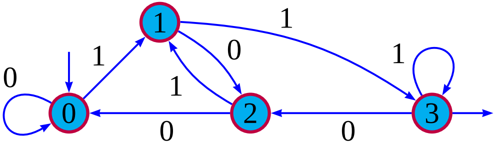
is accepted by a DFA that can be effectively constructed.
In particular, if $\mathbb{N}$ is U-recognizable, then any ultimately periodic set is U-recognizable. The proof is effective.

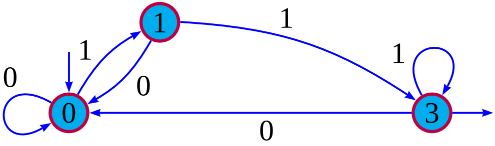


A DFA accepting $\mathrm{rep}(4\mathbb{N}+3)$.

Main question

What is the “best automaton” we can get ?





Question

In general, the derived algorithm does not provide a minimal automaton. What is the state complexity of the minimal automaton accepting $\mathrm{rep}_U(p\mathbb{N} + r)$?

BACKGROUND (I)

HONKALA'S DECISION PROCEDURE

Given any finite automaton recognizing a set X of integers written in base b. It is (algorithmically) decidable whether or not X is ultimately periodic.

- J. Honkala, A decision method for the recognizability of sets defined by number systems, *Theor. Inform. Appl.* **20** (1986).
- J.-P. Allouche, N. Rampersad, J. Shallit, Periodicity, repetitions, and orbits of an automatic sequence, *TCS* **410** (2009).
- J. P. Bell, É. Charlier, A. S. Fraenkel, M. R., A decision problem for ultimately periodic sets in non-standard numeration systems, *IJAC* **19** (2009).

Background (II)

Alexeev' result

Let $b, m \geq 2$. Let N, M be such that $b^N < m \leq b^{N+1}$ and

$$(m,1) < (m,b) < \cdots < (m,b^M) = (m,b^{M+1}) = (m,b^{M+2}) = \cdots.$$

The minimal automaton of $0^* \operatorname{rep}_b(m\mathbb{N})$ has exactly

$$\frac{m}{(m,b^{N+1})} + \sum_{t=0}^{\inf\{N,M-1\}} \frac{b^t}{(m,b^t)} \text{ states.}$$

- B. Alexeev, Minimal DFA for testing divisibility, *JCSS* **69** (2004).

Informations we are looking for

Consider a linear numeration system such that $\mathbb{N}$ is U-recognizable, how many states has the minimal automaton recognizing $0^* \operatorname{rep}_U(m\mathbb{N})$?

1. Give upper/lower bounds ?
2. Study special cases, e.g., Fibonacci numeration system ?
3. Get informations on the minimal automaton $\mathscr{A}_U$ recognizing $0^* \operatorname{rep}_U(\mathbb{N})$?

Structure of the minimal automaton $\mathscr{A}_U$ recognizing $0^* \operatorname{rep}_U(\mathbb{N})$

Bertrand numeration systems

- Bertrand numeration system: w is in $\operatorname{rep}_U(\mathbb{N})$ if and only if $w0$ is in $\operatorname{rep}_U(\mathbb{N})$.
- E.g., the Fibonacci system is Bertrand.

Theorem (Bertrand)

A system U is Bertrand if and only if there is a $\beta > 1$ such that

$$0^* \operatorname{rep}_U(\mathbb{N}) = \operatorname{Fact}(D_\beta).$$

Moreover, the system is derived from the β-development of 1.

If β is a Parry number, the system is linear and we have a minimal finite automaton $\mathscr{A}_\beta$ accepting $\operatorname{Fact}(D_\beta)$.

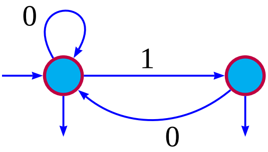
Theorem

Let U be a linear numeration system such that $\mathrm{rep}_U(\mathbb{N})$ is regular.

(i) The automaton $\mathscr{A}_U$ has a non-trivial strongly connected component $\mathscr{C}_U$ containing the initial state.

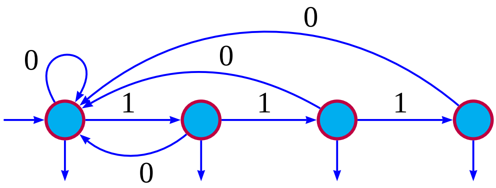
(ii) If p is a state in $\mathscr{C}_U$, then there exists $N \in \mathbb{N}$ such that $\delta_U(p, 0^n) = q_{U,0}$ for all $n \geq N$. In particular, one cannot leave $\mathscr{C}_U$ by reading a 0.

THE FIBONACCI NUMERATION SYSTEM



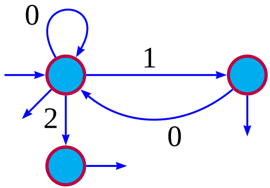
- $U_{n+2} = U_{n+1} + U_n$ ($U_0 = 1$, $U_1 = 2$)
- $\mathscr{A}_U$ accepts all words that do not contain 11.

The ℓ-bonacci numeration system



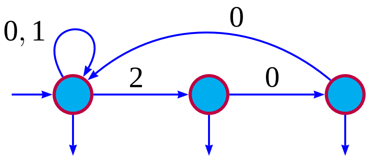
- $U_{n+\ell} = U_{n+\ell-1} + U_{n+\ell-2} + \cdots + U_n$
- $U_i = 2^i$, $i \in \{0, \ldots, \ell - 1\}$
- $\mathscr{A}_U$ accepts all words that do not contain 1^ℓ.

A NON-BERTRAND SYSTEM



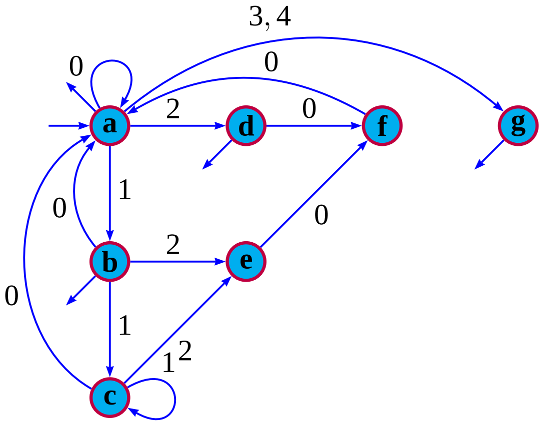
- $U_{n+2} = U_{n+1} + U_n, (U_0 = 1, U_1 = 3)$
- $(U_n)_{n \geq 0} = 1, 3, 4, 7, 11, 18, 29, 47, \dots$
- 2 is a greedy representation but 20 is not.

ANOTHER EXAMPLE



- Let β be the largest root of $X^3 - 2X^2 - 1$.
- $\mathsf{d}_\beta(1) = 2010^\omega$ and $\mathsf{d}^*_\beta(1) = (200)^\omega$.
- This automaton accepts $\mathrm{rep}_U(\mathbb{N})$ for U defined by $U_{n+3} = 2U_{n+2} + U_n, (U_0, U_1, U_2) = (1, 3, 7)$.

CHANGING THE INITIAL CONDITIONS



$U_{n+3} = 2U_{n+2} + U_n, (U_0, U_1, U_2) = (1, 3, 7)$
we change the initial values to $(U_0, U_1, U_2) = (1, 5, 6)$.

Theorem (cont'd.)

(iii) If $\mathscr{C}_U$ is the only non-trivial strongly connected component of $\mathscr{A}_U$, then $\lim_{n\to\infty} U_{n+1} - U_n = \infty$.

(iv) If $\lim_{n\to\infty} U_{n+1} - U_n = \infty$, then $\delta_U(q_{U,0}, 1)$ is in $\mathscr{C}_U$.

- U satisfies the dominant root condition if $\lim_{n\to\infty} U_{n+1}/U_n = \beta$ for some real $\beta > 1$.
- β is the dominant root of the recurrence.
- E.g., Fibonacci: dominant root $\beta = (1+\sqrt{5})/2$

Theorem (cont'd.)

Suppose U has a dominant root $\beta > 1$. If $\mathscr{A}_U$ has more than one non-trivial strongly connected component, then any such component other than $\mathscr{C}_U$ is a cycle all of whose edges are labeled 0.

An example with two components

- Let $t \geq 1$.
- Let $U_0 = 1$, $U_{tn+1} = 2U_{tn} + 1$, and
- $U_{tn+r} = 2U_{tn+r-1}$, for $1 < r \leq t$.
- E.g., for $t = 2$ we have $U = (1, 3, 6, 13, 26, 53, \ldots)$.
- Then $0^* \operatorname{rep}_U(\mathbb{N}) = \{0,1\}^* \cup \{0,1\}^* 2(0^t)^*$.
- The second component is a cycle of t 0's.

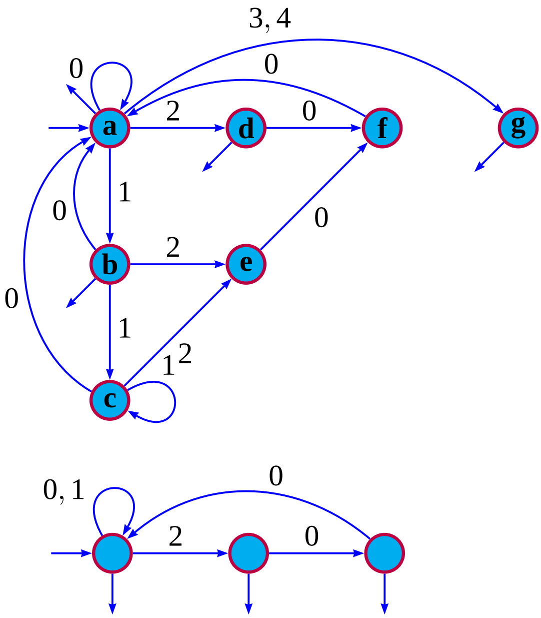
Relationship with $\mathscr{A}_\beta$

Theorem (cont'd.)

Suppose U has a dominant root $\beta > 1$. There is a morphism of automata Φ from $\mathscr{C}_U$ to $\mathscr{A}_\beta$.

Φ maps the states of $\mathscr{C}_U$ onto the states of $\mathscr{A}_\beta$ so that

- $\Phi(q_{U,0}) = q_{\beta,0}$,
- for all states q and all letters σ such that q and $\delta_U(q,\sigma)$ are in $\mathscr{C}_U$, we have $\Phi(\delta_U(q,\sigma)) = \delta_\beta(\Phi(q),\sigma)$.


3, 4
0
0
2
0
a
d
f
g
1
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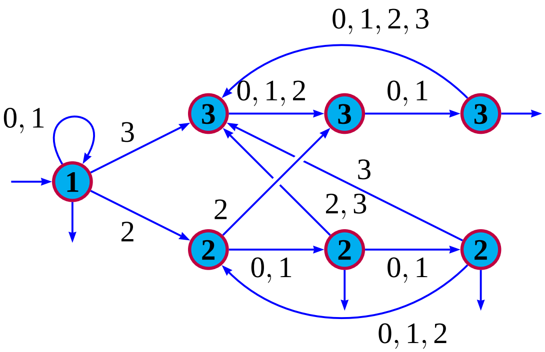
0
2
b
e
0
1
2
1
c
0, 1
0
2
0

Other results

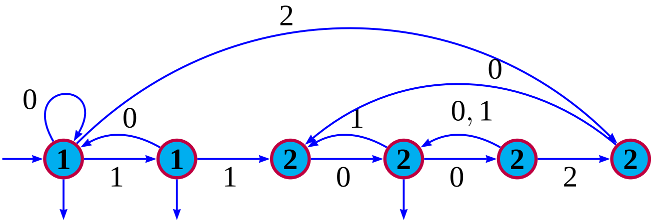
- When U has a dominant root $\beta > 1$, we can say more.
- E.g., if $\mathcal{A}_U$ has more than one strongly connected component, then $\mathrm{d}_\beta(1)$ is finite.
- We can also give sufficient conditions for $\mathscr{A}_U$ to have only one strongly connected component and sufficient conditions for $\mathscr{A}_U$ to have more than one strongly connected component.
- When U has no dominant root, the situation is more complicated.

A SYSTEM WITH NO DOMINANT ROOT



- $U_{n+3} = 24U_n, (U_0, U_1, U_2) = (1, 2, 6)$
- 3 strongly connected components

A SYSTEM WITH NO DOMINANT ROOT



- $U_{n+4} = 3U_{n+2} + U_n$, $(U_0, U_1, U_2, U_3) = (1, 2, 3, 7)$
- U_{n+1}/U_n does not converge, but
- $\lim_{n\to\infty} U_{2n+2}/U_{2n} = \lim_{n\to\infty} U_{2n+3}/U_{2n+1} = (3 + \sqrt{13})/2$

M. Hollander, Greedy numeration systems and regularity, *Theory Comput. Systems* **31** (1998).

Back to state complexity issues

The Hankel matrix

- Let $U = (U_n)_{n\geq 0}$ be a numeration system.
- For $t \geq 1$ define

$$H_t := \begin{pmatrix} U_0 & U_1 & \cdots & U_{t-1} \\ U_1 & U_2 & \cdots & U_t \\ \vdots & \vdots & \ddots & \vdots \\ U_{t-1} & U_t & \cdots & U_{2t-2} \end{pmatrix}.$$

- For $m \geq 2$, define $k_{U,m}$ to be the largest t such that $\det H_t \not\equiv 0 \pmod{m}$.

Calculating $k_{U,m}$

- $U_{n+2} = 2U_{n+1} + U_n$, $(U_0, U_1) = (1, 3)$
- $(U_n)_{n\geq 0} = 1, 3, 7, 17, 41, 99, 239, \ldots$
- $(U_n \bmod 2)_{n\geq 0}$ is constant and trivially satisfies the recurrence relation $U_{n+1} = U_n$ with $U_0 = 1$.
- Hence $k_{U,2} = 1$.
- Mod 4 we find $k_{U,4} = 2$.

A SYSTEM OF LINEAR CONGRUENCES

- Let $k = k_{U,m}$.
- Let $\mathbf{x} = (x_1, \ldots, x_k)$.
- Let $S_{U,m}$ denote the number of k-tuples $\mathbf{b}$ in $\{0, \ldots, m-1\}^k$ such that the system

$$H_k\,\mathbf{x} \equiv \mathbf{b} \pmod{m}$$

has at least one solution.

CALCULATING $S_{U,m}$

- $U_{n+2} = 2U_{n+1} + U_n$, $(U_0, U_1) = (1, 3)$
- Consider the system

$$\begin{cases} 1\,x_1 + 3\,x_2 & \equiv & b_1 \pmod 4 \\ 3\,x_1 + 7\,x_2 & \equiv & b_2 \pmod 4 \end{cases}$$

- $2x_1 \equiv b_2 - b_1 \pmod 4$
- For each value of b_1 there are at most 2 values for b_2.
- Hence $S_{U,4} = 8$.

Properties of the automata we consider

(H.1) $\mathscr{A}_U$ has a single strongly connected component $\mathscr{C}_U$.

(H.2) For all states p, q in $\mathscr{C}_U$ with $p \neq q$, there exists a word x_{pq} such that $\delta_U(p, x_{pq}) \in \mathscr{C}_U$ and $\delta_U(q, x_{pq}) \notin \mathscr{C}_U$, or vice-versa.

General state complexity result

Theorem

Let $m \geq 2$ be an integer. Let $U = (U_n)_{n\geq 0}$ be a linear numeration system such that

(a) $\mathbb{N}$ is U-recognizable and $\mathscr{A}_U$ satisfies (H.1) and (H.2),

(b) $(U_n \bmod m)_{n\geq 0}$ is purely periodic.

The number of states of the trim minimal automaton accepting $0^* \operatorname{rep}_U(m\mathbb{N})$ from which infinitely many words are accepted is

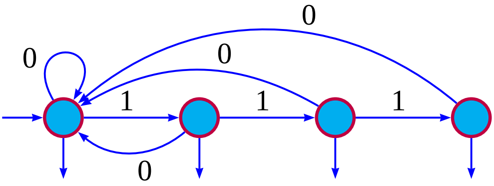
$$|\mathscr{C}_U| S_{U,m}.$$

Result for strongly connected automata

Corollary

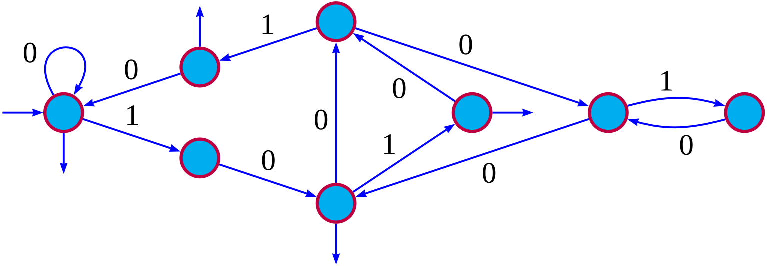
If U satisfies the conditions of the previous theorem and $\mathscr{A}_U$ is strongly connected, then the number of states of the trim minimal automaton accepting $0^* \operatorname{rep}_U(m\mathbb{N})$ is $|\mathscr{C}_U| S_{U,m}$.

RESULT FOR THE ℓ-BONACCI SYSTEM



COROLLARY

For U the ℓ-bonacci numeration system, the number of states of the trim minimal automaton accepting $0^* \operatorname{rep}_U(m\mathbb{N})$ is ℓm^ℓ.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Further work

- Analyze the structure of $\mathcal{A}_U$ for systems with no dominant root.
- Remove the assumption that U is purely periodic in the state complexity result.
- Big open problem: Given an automaton accepting $\mathrm{rep}_U(X)$, is it decidable whether X is an ultimately periodic set?

What I should not forget !