Overview of the $S$-adic Conjecture

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Basic notions and notations

\[ A = \{0, 1, \ldots, k - 1\} = \text{alphabet} \]
\[ w = w_0 w_1 w_2 \cdots \in A^\mathbb{N} \]
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Example

\( A = \{0, 1\} \)

\( w \) defined by

\[
  w_n = \begin{cases} 
    0 & \text{if } |\langle n \rangle_2|_0 \equiv 0 \pmod{2} \\
    1 & \text{if } |\langle n \rangle_2|_0 \equiv 1 \pmod{2} 
  \end{cases}
\]
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\[ w = 011010011001011010 \ldots \]

\[ w \text{ is called the } \text{Thue-Morse} \text{ sequence}. \]
Purely morphic sequences

Definition

\[ A^* = \bigcup_{n \in \mathbb{N}} A^n \]

\[ \sigma : A^* \rightarrow A^* \text{ is a morphism if } \forall u, v \in A^* \quad \sigma(uv) = \sigma(u)\sigma(v). \]
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Example

\[ \mu : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases} \]
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\end{cases} \]

\[
\begin{array}{|c|c|c|}
\hline
\mu(0) & \mu(01) & 01 \\
\mu^2(0) & \mu(01) & 0110 \\
\mu^3(0) & \mu(0110) & 01101001 \\
\hline
\end{array}
\]
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Example

\[ \begin{align*}
\mu : & \quad \begin{cases}
0 & \mapsto 01 \\
1 & \mapsto 10
\end{cases} \\
\mu(0) & = 01 \\
\mu^2(0) & = \mu(01) = 0110 \\
\mu^3(0) & = \mu(0110) = 01101001
\end{align*} \]

\( (\mu^n(0))_{n \in \mathbb{N}} \) converges to the Thue-Morse sequence.
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Definition

\( w \) is purely morphic if \( \exists \sigma \) s.t. \( w = \sigma^\omega(a) = \lim_{n \to +\infty} \sigma^n(a) \).
Factor complexity

Let $w \in A^N$.

Definition
The *complexity function* of $w$ is

$$p_w : \mathbb{N} \rightarrow \mathbb{N}, \quad n \mapsto \# \{ u \in A^n \mid u \text{ occurs in } w \}.$$
Factor complexity

Let \( w \in A^\mathbb{N} \).

**Definition**

The *complexity function* of \( w \) is

\[
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**Example (Thue-Morse sequence)**

\( w = 011010011001011010010110011010 \cdots \)
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\( w = 011010011001011010010110011010 \ldots \)

\( L_1(w) = \{0, 1\} \quad \Rightarrow \quad p_w(2) = 2 \)
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$L_2(w) = \{00, 01, 10, 11\}$ \quad $\Rightarrow$ \quad $p_w(2) = 4$
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L_1(w) &= \{0, 1\} & \Rightarrow & & p_w(2) = 2 \\
L_2(w) &= \{00, 01, 10, 11\} & \Rightarrow & & p_w(2) = 4 \\
L_3(w) &= \{001, 010, 011, 100, 101, 110\} & \Rightarrow & & p_w(2) = 6
\end{align*}
\]
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\]

\( \forall n \geq 2, \ p_w(n) \leq 4(n - 1). \)
Factor complexity of (purely) morphic sequences

Theorem (Pansiot)

Let \( w = \sigma^\omega(a) \) with \( \sigma \) non-erasing \((\sigma(a) \neq \varepsilon)\). Then,

\[
\rho_w(n) \in \{\Theta(1), \Theta(n), \Theta(n \log \log n), \Theta(n \log n), \Theta(n^2)\}.
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Theorem (Cobham - Pansiot)

If \( \sigma \) is erasing, then \( w = \phi(\tau^\omega(b)) \) with \( \tau \) non-erasing and \( \phi \) letter-to-letter.
Factor complexity of (purely) morphic sequences

Theorem (Pansiot)
Let $w = \sigma^\omega(a)$ with $\sigma$ non-erasing ($\sigma(a) \neq \varepsilon$). Then,

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Theorem (Cobham - Pansiot)
If $\sigma$ is erasing, then $w = \phi(\tau^\omega(b))$ with $\tau$ non-erasing and $\phi$ letter-to-letter.

Theorem (Deviatov)
If $w$ is morphic ($w = \phi(\tau^\omega(b))$), then either $\exists k \in \mathbb{N}^* \text{ s.t. } p_w \in \Theta(n^{1+\frac{1}{k}})$ or $p_w \in O(n \log n)$. 
General properties of $\rho_w$

$w \in A^N$
General properties of $\rho_w$

$w \in A^\mathbb{N}$

$\forall n \quad 1 \leq \rho_w(n) \leq (\#A)^n$;
General properties of $p_w$

$w \in A^\mathbb{N}$

- $\forall n \quad 1 \leq p_w(n) \leq (\#A)^n$;
- $\forall m, n \quad p_w(m + n) \leq p_w(m)p_w(n)$;
General properties of $\rho_w$

$w \in A^\mathbb{N}$

- $\forall n \quad 1 \leq \rho_w(n) \leq (\#A)^n$;
- $\forall m, n \quad \rho_w(m + n) \leq \rho_w(m)\rho_w(n)$;
- $\rho_w$ is non-decreasing.
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- $\forall n \quad 1 \leq \rho_w(n) \leq (#A)^n$;
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- $\rho_w$ is non-decreasing.

Theorem (Morse-Hedlund)

The following are equivalent:
1. $w$ is ultimately periodic, i.e., $w = uvvvv \cdots$;
2. $\exists n_0 \quad \rho_w(n_0) \leq n_0$;
3. $\rho_w$ is ultimately constant.
Sturmian sequences

Sturmian sequences are binary sequences $w$ satisfying $\rho_w(n) = n + 1$ for all $n$. 
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Define the 4 substitutions:

$L_0 : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \end{cases}$

$R_0 : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 10 \end{cases}$

$L_1 : \begin{cases} 0 \mapsto 10 \\ 1 \mapsto 1 \end{cases}$

$R_1 : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1 \end{cases}$
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Define the 4 substitutions:

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$L_1 : \begin{cases} 0 \mapsto 10 \\ 1 \mapsto 1 \end{cases} \quad R_1 : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1 \end{cases}$

Then

$$w = L_0^{n_1} R_0^{n_2} L_1^{n_3} R_1^{n_4} L_0^{n_5} R_0^{n_6} \ldots (0).$$
S-adicity

Definition

\( w \in A^\mathbb{N} \) is \textit{S-adic} if there are

- a finite set of non-erasing morphisms \( \sigma \);
- a sequence \(( \sigma_n : A_{n+1}^* \to A_n^* )_n\) in \( S^\mathbb{N} \);
- a sequence \((a_n \in A_n)_n\) of letters;

such that:

\[
  w = \lim_{n \to +\infty} \sigma_0 \sigma_1 \cdots \sigma_n(a_{n+1}).
\]
S-adicity

Definition

\( w \in A^\mathbb{N} \) is **S-adic** if there are

- \( S = \) finite set of non-erasing morphisms \( \sigma \);
- a sequence \( (\sigma_n : A_{n+1}^* \rightarrow A_n^*)_n \) in \( S^\mathbb{N} \);
- a sequence \( (a_n \in A_n)_n \) of letters;

such that:

\[
    w = \lim_{n \to +\infty} \sigma_0 \sigma_1 \cdots \sigma_n(a_{n+1}).
\]

Remark

If \( \sigma_n(a_{n+1}) = x_1 x_2 x_3 \):

\[
    \sigma_0 \cdots \sigma_n(a_{n+1}) = \sigma_0 \cdots \sigma_{n-1}(x_1 x_2 x_3) = \sigma_0 \cdots \sigma_{n-1}(x_1) \sigma_0 \cdots \sigma_{n-1}(x_2) \sigma_0 \cdots \sigma_{n-1}(x_3)
\]
S-adicity

Example

\[ \varphi : \begin{cases} 
0 & \mapsto 01 \\
1 & \mapsto 0 
\end{cases} \]

\[ \mu : \begin{cases} 
0 & \mapsto 01 \\
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\[ w = \lim_{n \to +\infty} \varphi \mu \varphi^2 \mu^2 \cdots \varphi^n \mu^n(0) \]
S-adicity

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\[ \varphi \mu (0) = \begin{bmatrix} \varphi (0) \\ \varphi (1) \end{bmatrix} = \begin{bmatrix} 01 \\ 0 \end{bmatrix} \]

\[ \varphi \mu \varphi^2 (0) = \begin{bmatrix} \varphi \mu (0) \\ \varphi \mu (1) \\ \varphi \mu (0) \end{bmatrix} = \begin{bmatrix} 010 \\ 001 \\ 010 \end{bmatrix} \]
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\[ \varphi \mu(0) = \begin{array}{c|c} \varphi(0) & \varphi(1) \end{array} = \begin{array}{c} 01 \end{array} \]

\[ \varphi \mu \varphi^2(0) = \begin{array}{c|c|c} \varphi \mu(0) & \varphi \mu(1) & \varphi \mu(0) \end{array} = \begin{array}{c|c|c} 01 & 01 & 01 \end{array} \]

\[ (\varphi \mu \varphi^2 \mu^2 \cdots \varphi^n \mu^n(0))_n \text{ converges in } A^\mathbb{N}. \]
Many classical examples are $S$-adic
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- ultimately periodic sequences are *morphic*, i.e.,
  \[ w = \tau(\sigma^\omega(a)); \]
Many classical examples are $S$-adic

- ultimately periodic sequences are *morphic*, i.e., $w = \tau(\sigma^\omega(a))$;
- $k$-*automatic sequences* are morphic with $\sigma$ $k$-uniform and $\tau$ letter-to-letter;
Many classical examples are \( S \)-adic

- ultimately periodic sequences are *morphic*, i.e., \( w = \tau(\sigma^\omega(a)) \);
- \( k \)-automatic sequences are morphic with \( \sigma \) \( k \)-uniform and \( \tau \) letter-to-letter;
- *codings of rotations* of parameters \( (\alpha, \beta) \) are \( S \)-adic (with \( \text{Card}(S) = 5) \);
Many classical examples are $S$-adic

A sequence $\mathbf{w}$ is Sturmian if and only $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $x \in [0, 1]$ such that

$$w_k = \begin{cases} 0 & \text{if } R_\alpha^k(x) \in [0, 1 - \alpha[ \\ 1 & \text{if } R_\alpha^k(x) \in [1 - \alpha, 1[ \end{cases}$$
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Rotations of parameters \((\alpha, \beta)\):
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- *\( k \)-automatic sequences* are morphic with \( \sigma \) \( k \)-uniform and \( \tau \) letter-to-letter;
- *codings of rotations* of parameters \((\alpha, \beta)\) are S-adic (with \( \text{Card}(S) = 5 \));
- *codings of 3-IET* are S-adic (with \( \text{Card}(S) = 5 \));
Many classical examples are $S$-adic

Sturmian rotation view as an IET:

$$R_\alpha$$
Many classical examples are S-adic

Sturmian case:

\[ R_\alpha \]

\[ T \]

3-IET:
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- $k$-automatic sequences are morphic with $\sigma$ $k$-uniform and $\tau$ letter-to-letter;
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- *codings of 3-IET* are $S$-adic (with $\text{Card}(S) = 5$);
- *Arnoux-Rauzy sequences* are $S$-adic (with $\text{Card}(S) = \text{Card}(A)$);
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- *Arnoux-Rauzy sequences* are S-adic (with $\text{Card}(S) = \text{Card}(A)$);
- *episturmian sequence* are S-adic (with $\text{Card}(S) = 2\text{Card}(A)$);
Many classical examples are S-adic

- ultimately periodic sequences are morphic, i.e., \( w = \tau(\sigma^w(a)) \);
- \( k \)-automatic sequences are morphic with \( \sigma \) \( k \)-uniform and \( \tau \) letter-to-letter;
- codings of rotations of parameters \((\alpha, \beta)\) are S-adic (with \( \text{Card}(S) = 5 \));
- codings of 3-IET are S-adic (with \( \text{Card}(S) = 5 \));
- Arnoux-Rauzy sequences are S-adic (with \( \text{Card}(S) = \text{Card}(A) \));
- episturmian sequence are S-adic (with \( \text{Card}(S) = 2\text{Card}(A) \));
- linearly recurrent sequences;
S-adic conjecture

All examples previously cited have a sub-linear complexity \( \rho(n) \leq Cn \)

**Question:** Is there any relation between "S-adic" and "low complexity"?
**S-adic conjecture**

All examples previously cited have a sub-linear complexity \( \rho(n) \leq Cn \)

**Question:** Is there any relation between "S-adic" and "low complexity"?

**Conjecture**

*There is a condition C such that w has an at most linear complexity if and only if it is a S-adic sequence satisfying the condition C.*
We cannot avoid $C$

For $\text{Card}(S) = 1$, it is possible to get a quadratic complexity (Pansiot).
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For $\text{Card}(S) = 1$, it is possible to get a quadratic complexity (Pansiot).

**Proposition (Cassaigne)**

$A = \text{alphabet}$.  
$\exists S$ with $\text{Card}(S) = \text{Card}(A) + 1$ s.t. any $w \in A^\mathbb{N}$ is $S$-adic.
We cannot avoid $C$

For $\text{Card}(S) = 1$, it is possible to get a quadratic complexity (Pansiot).

**Proposition (Cassaigne)**

$A = \text{alphabet}.
\exists S \text{ with } \text{Card}(S) = \text{Card}(A) + 1 \text{ s.t. any } w \in A^\mathbb{N} \text{ is } S\text{-adic.}$

**Proof.**

$l \notin A.$

$\forall a \in A : \sigma_a : \begin{cases} \ell \mapsto \ell a \\ b \mapsto b \ \forall b \neq \ell \end{cases}$

$\phi : \begin{cases} \ell \mapsto w_0 \\ b \mapsto b \ \forall b \neq \ell \end{cases}$

$w = \phi \sigma_{w_2} \sigma_{w_3} \sigma_{w_4} \sigma_{w_5}(\ell) \cdots$
Example of condition $C$

**Theorem (Durand)**

$w$ is linearly recurrent if and only if it is a primitive and proper $S$-adic sequence with $\text{Card}(S) < \infty$. 
Example of condition $C$

Theorem (Durand)

$w$ is linearly recurrent if and only if it is a primitive and proper $S$-adic sequence with $\text{Card}(S) < \infty$.

Primitive $S$-adic: $\exists s_0$ such that $\forall r, \forall b \in A_r, c \in A_{r+s_0+1}$: $b$ occurs in $\sigma_r \sigma_{r+1} \cdots \sigma_{r+s_0}(c)$. 
Example of condition \( C \)

**Theorem (Durand)**

\( w \) is linearly recurrent if and only if it is a primitive and proper \( S \)-adic sequence with \( \text{Card}(S) < \infty \).

*Primitive \( S \)-adic:* \( \exists s_0 \) such that \( \forall r, \forall b \in A_r, c \in A_{r+s_0+1} : 
\quad b \text{ occurs in } \sigma_r \sigma_{r+1} \cdots \sigma_{r+s_0}(c). \)

*Proper \( S \)-adic:* \( \forall \sigma \in S, \exists b, c \in A \) such that \( \sigma(a) \in bA^*c \ \forall a \in A. \)
First difficulty to find $C$: the growth rate of the images

For one morphism $\sigma$:

$$|\sigma^n(a)| \in \Theta(n^\alpha \beta^n)$$

and the complexity only depends on

$$\max_{a,b} \frac{|\sigma^n(a)|}{|\sigma^n(b)|}$$
First difficulty to find $C$: the growth rate of the images

For one morphism $\sigma$:

$$|\sigma^n(a)| \in \Theta(n^\alpha \beta^n)$$

and the complexity only depends on

$$\max_{a,b} \frac{|\sigma^n(a)|}{|\sigma^n(b)|}$$

For $S$-adic sequences:

$$|\sigma^n| \sim |\sigma_0 \sigma_1 \cdots \sigma_n|$$

but $|\sigma_0 \sigma_1 \cdots \sigma_n|$ does not have any analytic description.
Generalizing Pansiot’s conditions only provides a sufficient condition

Theorem (Pansiot)

*If* \( \min_a |\sigma^n(a)| \to +\infty \), *then*

\[
p(n) \leq Kn \iff \max_{a,b} \frac{|\sigma^n(a)|}{|\sigma^n(b)|}.
\]
Generalizing Pansiot’s conditions only provides a sufficient condition

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Theorem (Durand)

If \( \min_a |\sigma_0\sigma_1\cdots\sigma_n(a)| \to +\infty \), then

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p(n) \leq Kn \iff \max_{a, b} \left| \frac{|\sigma_0\sigma_1\cdots\sigma_n(a)|}{|\sigma_0\sigma_1\cdots\sigma_n(b)|} \right|.
\]

Sturmian
There are some good sets $S$

**Corollary**

*If $S$ contains only strongly primitive or uniform morphisms, then*

$$p(n) \leq Kn.$$
There are some good sets $S$

**Corollary**

*If $S$ contains only strongly primitive or uniform morphisms, then*

$$p(n) \leq Kn.$$  

**Proposition**

*If $S = \{\varphi, \mu\}$ with*

$$
\varphi : \begin{cases} 
0 \mapsto 01 \\
1 \mapsto 0
\end{cases} 
\quad \mu : \begin{cases} 
0 \mapsto 01 \\
1 \mapsto 10
\end{cases},
$$

*then any $S$-adic sequence has a sub-linear complexity.*
Naive idea 1: take only "good morphisms" yields to sub-linear complexity

Conjecture (Boshernitzan)

If $S$ is finite and contains only morphisms that creates sub-linear complexity, then any $S$-adic sequence has a sub-linear complexity.
Naive idea 1: take only "good morphisms" yields to sub-linear complexity

Conjecture (Boshernitzan)

*If S is finite and contains only morphisms that creates sub-linear complexity, then any S-adic sequence has a sub-linear complexity.*

Counter-Example (Boshernitzan)

Let

\[
\alpha : \begin{cases} 
    a \mapsto aab \\
    b \mapsto b
\end{cases} \quad \text{and} \quad E : \begin{cases} 
    a \mapsto b \\
    b \mapsto a
\end{cases}
\]

and consider

\[
w = \lim_{n \to \infty} \alpha E \alpha^2 E \alpha^3 E \cdots \alpha^{n-1} E \alpha^n (aaa \cdots ).
\]
Naive idea 1: take only "good morphisms" yields to sub-linear complexity

Conjecture (Boshernitzan)

If $S$ is finite and contains only morphisms that creates sub-linear complexity, then any $S$-adic sequence has a sub-linear complexity.

Counter-Example (Boshernitzan)

Let $S = \{\alpha E, E\alpha\}$. We have

\[
\begin{align*}
\alpha E : & \quad \begin{cases} a \mapsto b \\ b \mapsto aab \end{cases} \\
E\alpha : & \quad \begin{cases} a \mapsto bba \\ b \mapsto a \end{cases}
\end{align*}
\]
Naive idea 1: take only "good morphisms" yields to sub-linear complexity

Conjecture (Boshernitzan)

If $S$ is finite and contains only morphisms that creates sub-linear complexity, then any $S$-adic sequence has a sub-linear complexity.

Counter-Example (Boshernitzan)
Let $S = \{αE, Eα\}$. We have

$$αE : \begin{cases} a \mapsto b \\ b \mapsto aab \end{cases} \quad \text{and} \quad Eα : \begin{cases} a \mapsto bba \\ b \mapsto a \end{cases}$$

► $w$ is $S$-adic:

$$αEα²Eα³Eα⁴E \ldots$$

$$(αE)(αE)(Eα)(Eα)(αE)(Eα)(Eα)(αE)(Eα)(αE) \ldots$$
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- $w$ is $S$-adic:

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Counter-Example (Boshernitzan)

Let $S = \{\alpha E, E\alpha\}$. We have

$\alpha E : \begin{cases} a \mapsto b \\ b \mapsto aab \end{cases}$ and $E\alpha : \begin{cases} a \mapsto bba \\ b \mapsto a \end{cases}$

- $w$ is $S$-adic:
- $\alpha E$ and $E\alpha$ are primitive;
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\end{align*}
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- $w$ is $S$-adic:
- $\alpha E$ and $E\alpha$ are primitive;
- $w$ does not have a sub-linear complexity.
Naive idea 2: a "bad morphism" in $S$ creates too much complexity.
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Counter-Example 1

$\alpha : \begin{cases} a \mapsto aab \\ b \mapsto b \end{cases}$ and $\mu : \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}$

$w = \lim_{n \to \infty} \alpha^k \mu \alpha^k \mu \alpha^k \mu \cdots \alpha^k \mu \alpha^k \mu \alpha^k \mu \alpha \mu (aaa \cdots)$. 
Naive idea 2: a "bad morphism" in $S$ creates too much complexity

Counter-Example 1

$$\alpha : \begin{cases} a \mapsto aab \\ b \mapsto b \end{cases} \quad \text{and} \quad \mu : \begin{cases} a \mapsto ab \\ b \mapsto ba \end{cases}$$

$$w = \lim_{n \to \infty} \alpha^{k_0} \mu \alpha^{k_1} \mu \alpha^{k_2} \mu \cdots \alpha^{k_{n-1}} \mu \alpha^{k_n}(aaa \cdots).$$

$\Rightarrow \alpha^\omega(a)$ has a quadratic complexity;
Naive idea 2: a "bad morphism" in $S$ creates too much complexity

Counter-Example 1

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- $\alpha^\omega (a)$ has a quadratic complexity;
- $w$ has a sub-linear complexity if and only if $(k_n)_{n \in \mathbb{N}}$ is bounded.
Naive idea 2: a "bad morphism" in $S$ creates too much complexity

Counter-Example 2

$$
\beta : \begin{cases}
a \mapsto aab \\
b \mapsto bbc \\
c \mapsto c
\end{cases}
\quad \text{and} \quad
M : \begin{cases}
a \mapsto a \\
b \mapsto b \\
c \mapsto b
\end{cases}
$$

$$
\mathbf{w} = \lim_{n \to \infty} M\beta M\beta^2 M\beta^3 M \cdots \beta^{n-1} M\beta^n(aaa \cdots)
$$
Naive idea 2: a "bad morphism" in $S$ creates too much complexity

Counter-Example 2

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$$w = \lim_{n \to \infty} M\beta M\beta^2 M\beta^3 M \cdots \beta^{n-1} M\beta^n (aaa \cdots).$$

- $\beta^\omega (a)$ has a quadratic complexity and occurs in arbitrary long ranges in the sequence of morphisms;
Naive idea 2: a "bad morphism" in $S$ creates too much complexity

Counter-Example 2

$$\beta : \begin{cases} a \mapsto aab \\ b \mapsto bbc \\ c \mapsto c \end{cases} \text{ and } M : \begin{cases} a \mapsto a \\ b \mapsto b \\ c \mapsto b \end{cases}$$

$$w = \lim_{n \to \infty} M\beta M\beta^2 M\beta^3 M \cdots \beta^{n-1} M\beta^n(aaa \cdots).$$

- $\beta^\omega(a)$ has a quadratic complexity and occurs in arbitrary long ranges in the sequence of morphisms;
- $w$ has a sub-linear complexity.
Conclusions

1. There are some "good sets" $S$: all sequences $(\sigma_n)_{n \in \mathbb{N}} \in S^\mathbb{N}$ create sub-linear complexity
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2. For some sets $S$, the sequence of morphisms is really important (counter-example 1).
Conclusions

1. There are some "good sets" $S$: all sequences $(\sigma_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}}$ create sub-linear complexity

2. For some sets $S$, the sequence of morphisms is really important (counter-example 1)

3. Even if $S$ contains only "good morphisms", there might be some $S$-adic sequences with high complexity (Boshernitzan’s counter-example)
Conclusions

1. There are some "good sets" $S$: all sequences $(\sigma_n)_{n \in \mathbb{N}} \in S^\mathbb{N}$ create sub-linear complexity

2. For some sets $S$, the sequence of morphisms is really important (counter-example 1)

3. Even if $S$ contains only "good morphisms", there might be some $S$-adic sequences with high complexity (Boshernitzan’s counter-example)

4. Even when a "bad morphism" occurs very often in $(\sigma_n)_{n \in \mathbb{N}}$, the complexity can be sub-linear (counter-example 2)
Thank you