Diagonalisation (quelques exemples)

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Soit la matrice

$$A = \begin{pmatrix} 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

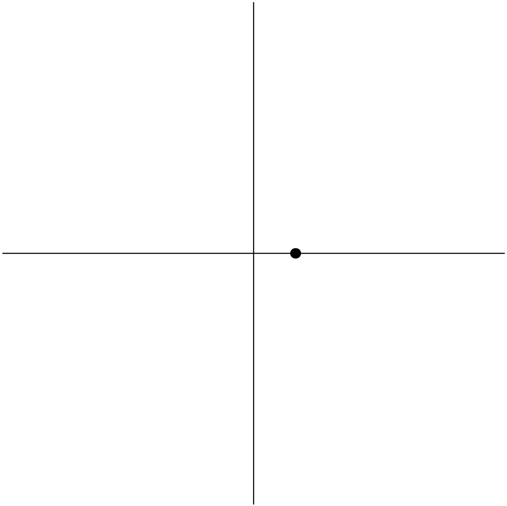
On va regarder

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots, A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots$$

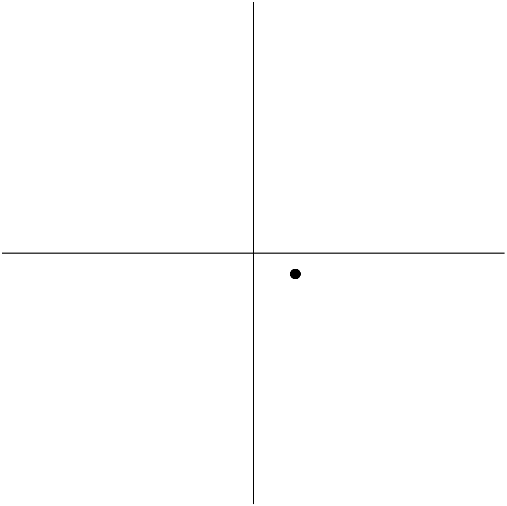
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \dots, A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \dots$$

$$\begin{pmatrix} -2 \\ -1 \end{pmatrix}, \dots, A^n \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \dots$$

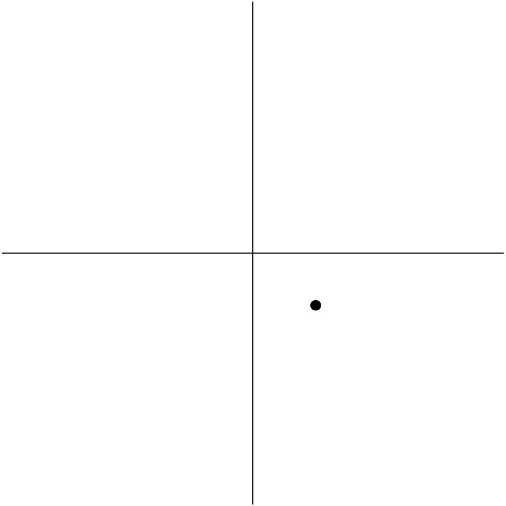
$$A^0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



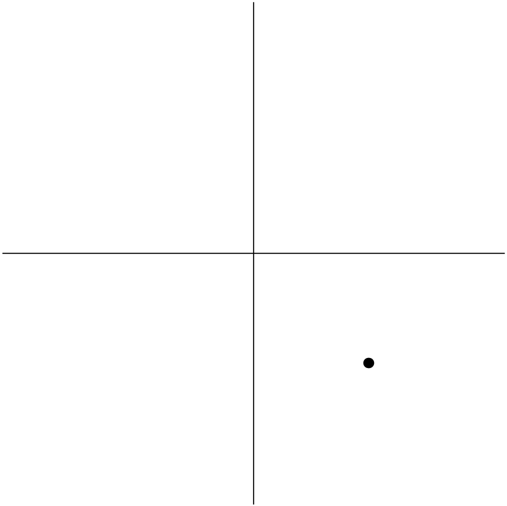
$$A^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



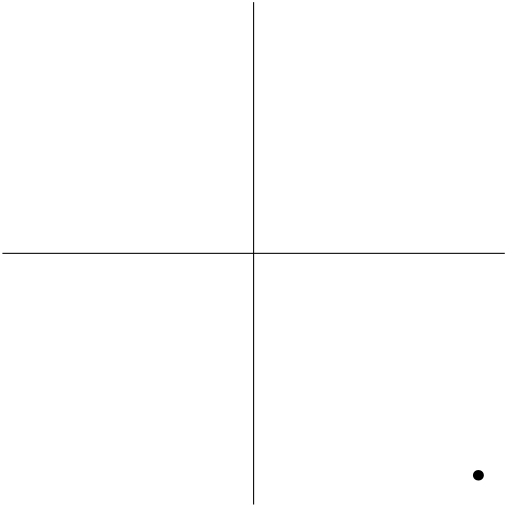
$$A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



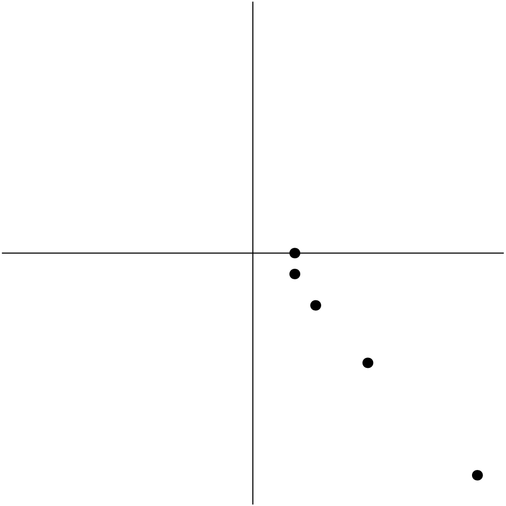
$$A^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



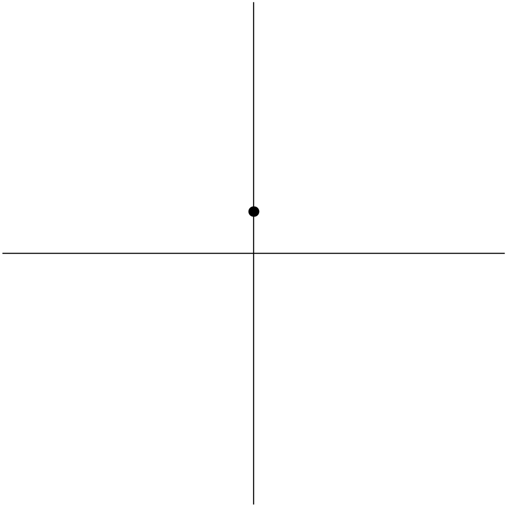
$$A^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



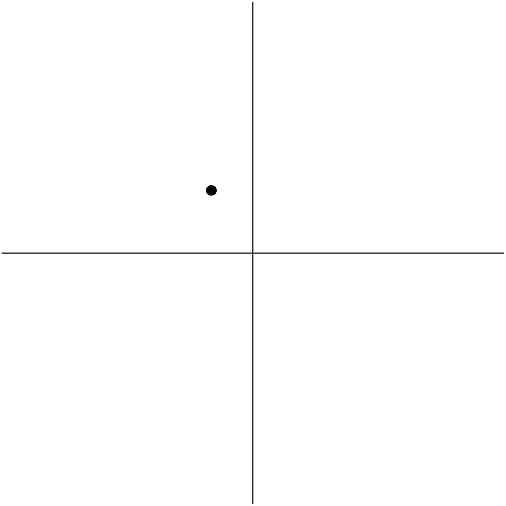
$$\begin{pmatrix}1\\0\end{pmatrix}, \ldots, A^4\begin{pmatrix}1\\0\end{pmatrix}$$



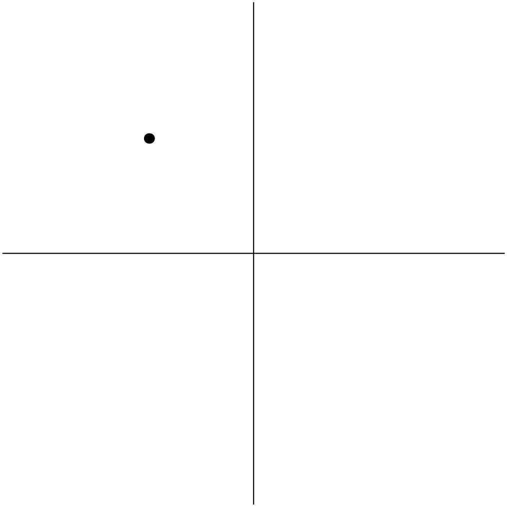
$$A^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



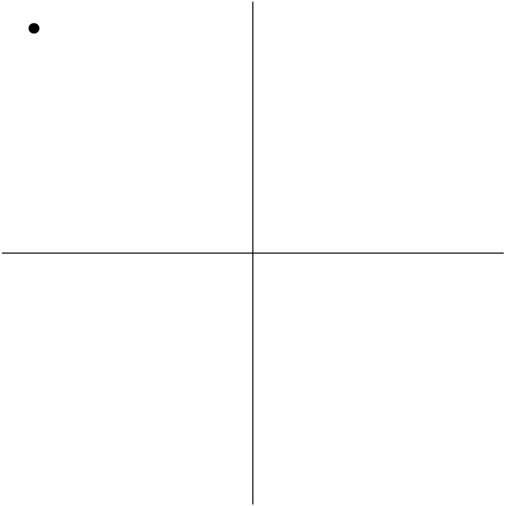
$$A^1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



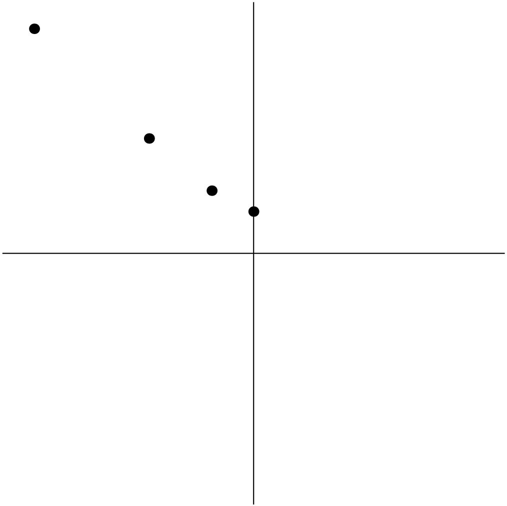
$$A^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



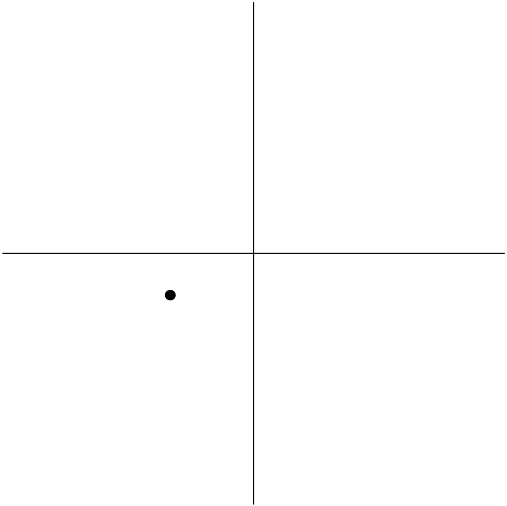
$$A^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



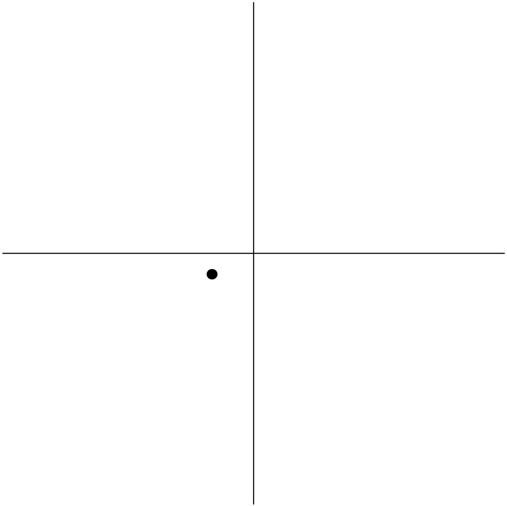
$$\begin{pmatrix}0\\1\end{pmatrix}, \ldots, A^3\begin{pmatrix}0\\1\end{pmatrix}$$



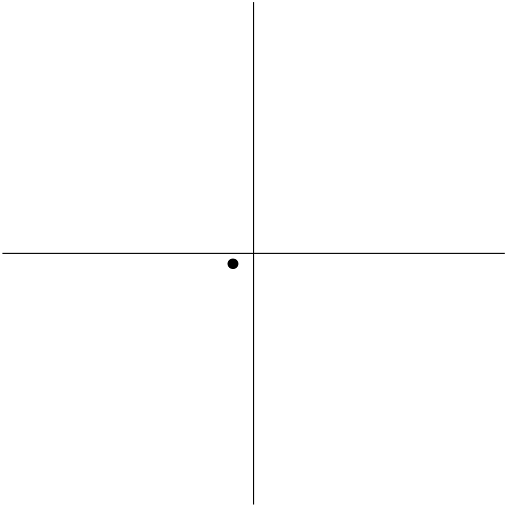
$$A^0 \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$



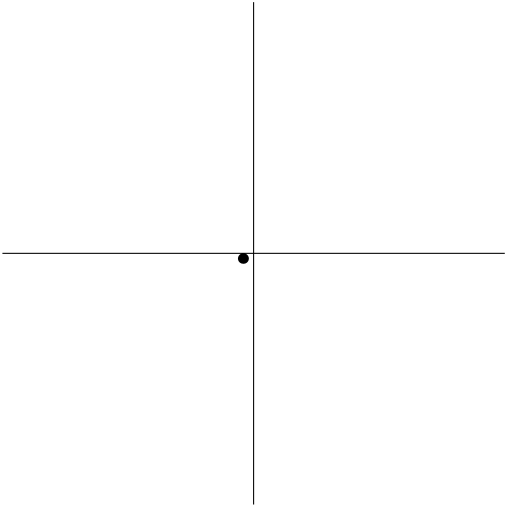
$$A^1 \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$



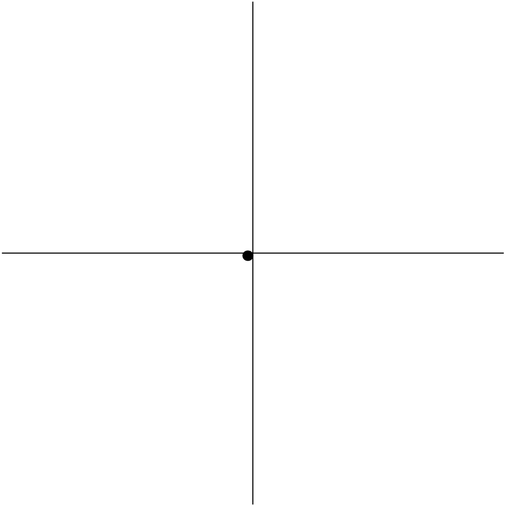
$$A^2 \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$



$$A^3 \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$



$$A^4 \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$



$$\det(A-\lambda\, I) = \begin{pmatrix} 1-\lambda & -1 \\ -\frac{1}{2} & \frac{3}{2}-\lambda \end{pmatrix} = \lambda^2 - \frac{5}{2}\lambda + 1 = (\lambda-2)\left(\lambda - \frac{1}{2}\right).$$

Deux valeurs propres simples : $\frac{1}{2}$ et 2

Espace propre associé à $1/2$

$$\left(A - \frac{1}{2} I\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow \left\{ \begin{array}{rcl} \frac{1}{2}x_1 - x_2 & = & 0 \\ -\frac{1}{2}x_1 + x_2 & = & 0 \end{array} \right.$$

$$x_1 = 2x_2$$

$$E_{1/2} = \rangle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \langle$$

Espace propre associé à 2

$$(A-2I)\begin{pmatrix}x_1\\x_2\end{pmatrix}=0 \Leftrightarrow \left\{\begin{array}{rcl} -x_1-x_2 & = & 0\\ -\frac{1}{2}x_1-\frac{1}{2}x_2 & = & 0\end{array}\right.$$

$$x_1=-x_2$$

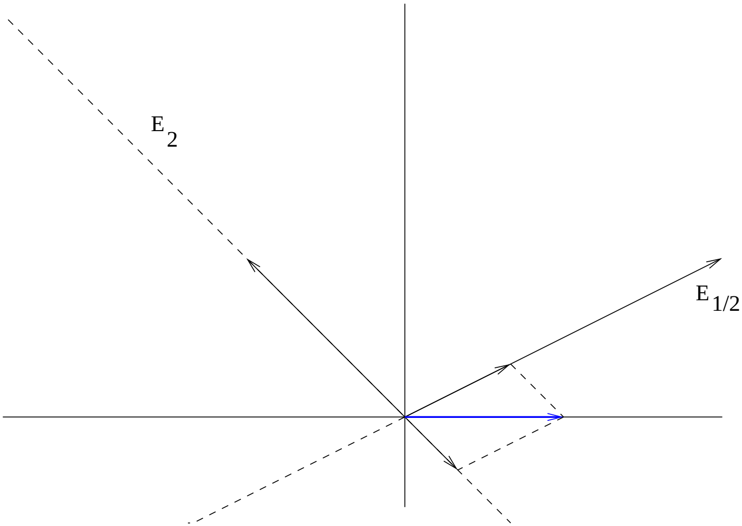
$$E_2=\rangle\begin{pmatrix}-1\\1\end{pmatrix}\langle$$

A (vu comme endomorphisme de $\mathbb{R}^2$) est diagonalisable donc

$$\mathbb{R}^2 = E_{1/2} \oplus E_2 = \rangle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \langle \oplus \rangle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \langle$$

On dispose d'une base de $\mathbb{R}^2$ formée de vecteurs propres de A. Ainsi,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-\frac{1}{3}) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix} + \left(-\frac{1}{3}\right)\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix} + \left(-\frac{1}{3}\right)\begin{pmatrix}-1\\1\end{pmatrix}$$

$$A\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}A\underbrace{\begin{pmatrix}2\\1\end{pmatrix}}_{\in E_{1/2}} + \left(-\frac{1}{3}\right)A\underbrace{\begin{pmatrix}-1\\1\end{pmatrix}}_{\in E_2}$$

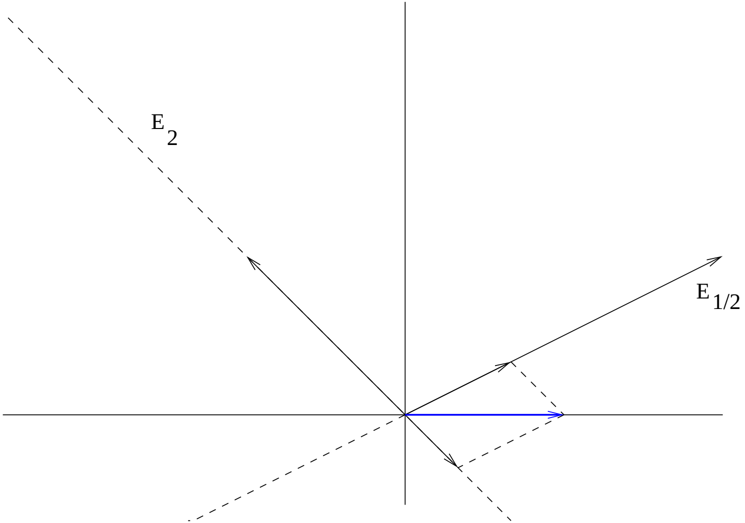
$$A\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}\,\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix} + \left(-\frac{1}{3}\right)2\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix} + (-\frac{1}{3})\begin{pmatrix}-1\\1\end{pmatrix}$$

$$A\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}A\underbrace{\begin{pmatrix}2\\1\end{pmatrix}}_{\in E_{1/2}} + (-\frac{1}{3})A\underbrace{\begin{pmatrix}-1\\1\end{pmatrix}}_{\in E_2}$$

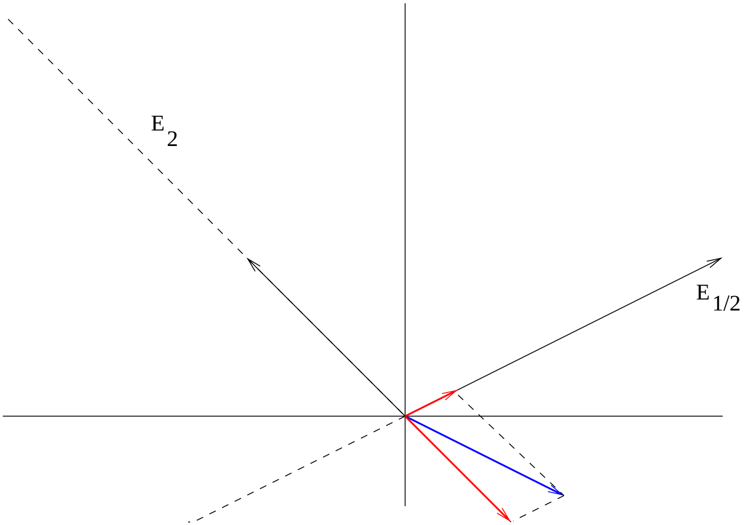
$$A\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}\,\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix} + (-\frac{1}{3})\,2\begin{pmatrix}-1\\1\end{pmatrix}$$

Avant application de A



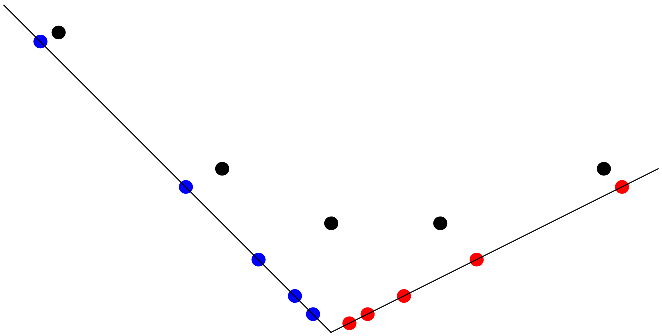
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \left(-\frac{1}{3}\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Après application de A

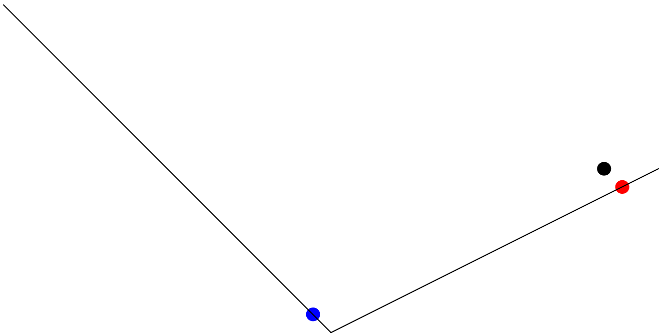


$$A\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{3}\,\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix} + (-\frac{1}{3})\,2\begin{pmatrix}-1\\1\end{pmatrix}$$

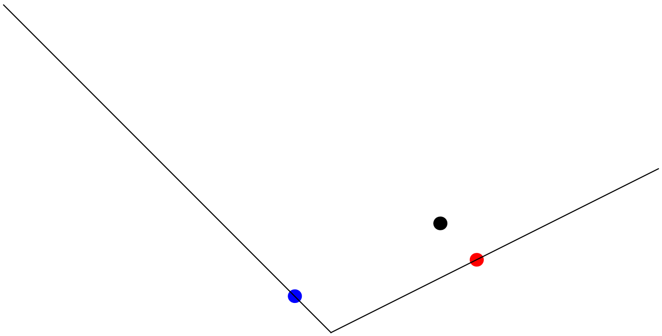
$$8\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}$$



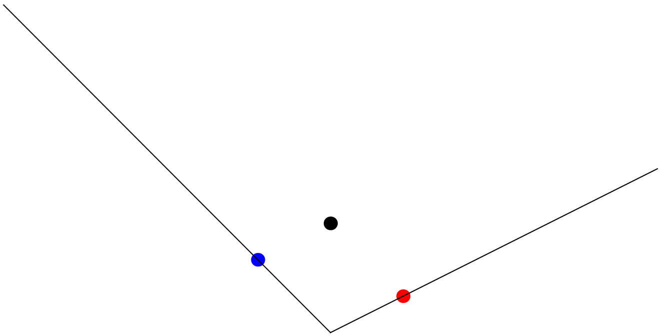
$$8\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}$$



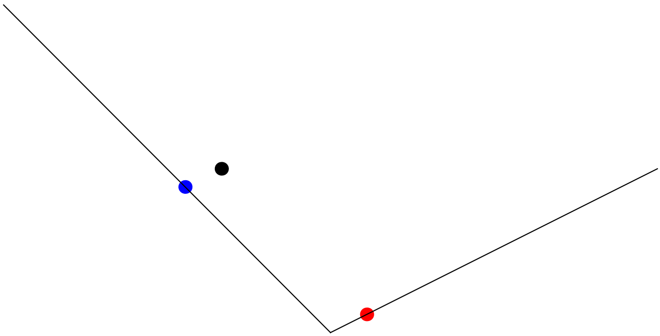
$$A(8\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}) = 8A\begin{pmatrix}2\\1\end{pmatrix}+A\begin{pmatrix}-1\\1\end{pmatrix}$$



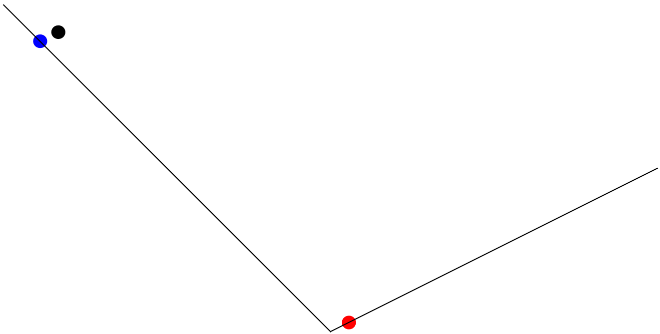
$$A^2(8\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}) = 8A^2\begin{pmatrix}2\\1\end{pmatrix}+A^2\begin{pmatrix}-1\\1\end{pmatrix}$$



$$A^3(8\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}) = 8A^3\begin{pmatrix}2\\1\end{pmatrix}+A^3\begin{pmatrix}-1\\1\end{pmatrix}$$

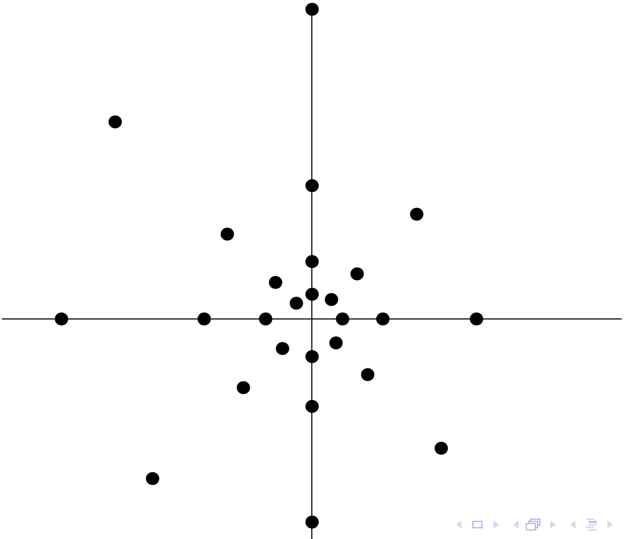


$$A^4(8\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}-1\\1\end{pmatrix}) = 8A^4\begin{pmatrix}2\\1\end{pmatrix}+A^4\begin{pmatrix}-1\\1\end{pmatrix}$$



Une matrice à coefficients réels peut avoir des vecteurs/valeurs propres non réels,

$$R = \frac{9}{10} \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$$



$$R = \frac{9}{10}\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$$

$$\det(R - \lambda I) = \lambda^2 - \frac{9\sqrt{2}}{10}\lambda + \frac{81}{100} = \left(\lambda - \frac{9\sqrt{2}}{20}(1+i)\right)\left(\lambda - \frac{9\sqrt{2}}{20}(1-i)\right)$$

Deux valeurs propres simples (conjuguées car $\chi_A \in \mathbb{R}[\lambda]$) :
$\frac{9\sqrt{2}}{20}(1+i) = \frac{9}{10}e^{i\pi/4}$ et $\frac{9\sqrt{2}}{20}(1-i) = \frac{9}{10}e^{-i\pi/4}$

$$E_{\frac{9\sqrt{2}}{20}(1+i)} = \rangle \begin{pmatrix} i \\ 1 \end{pmatrix} \langle, \quad E_{\frac{9\sqrt{2}}{20}(1-i)} = \rangle \begin{pmatrix} -i \\ 1 \end{pmatrix} \langle$$

$$S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \text{ et } S^{-1}RS = \begin{pmatrix} \frac{9\sqrt{2}}{20}(1+i) & 0 \\ 0 & \frac{9\sqrt{2}}{20}(1-i) \end{pmatrix}$$

$$\mathbb{C}^2 = \rangle \begin{pmatrix} i \\ 1 \end{pmatrix} \langle \oplus \rangle \begin{pmatrix} -i \\ 1 \end{pmatrix} \langle$$

$\forall a, b \in \mathbb{C}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{b - ai}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{b + ai}{2} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

et

$$R \begin{pmatrix} a \\ b \end{pmatrix} = \frac{b - ai}{2} \frac{9}{10} e^{i\pi/4} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{b + ai}{2} \frac{9}{10} e^{-i\pi/4} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$R^n \begin{pmatrix} a \\ b \end{pmatrix} = \frac{b - ai}{2} \left(\frac{9}{10} e^{i\pi/4} \right)^n \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{b + ai}{2} \left(\frac{9}{10} e^{-i\pi/4} \right)^n \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

En particulier, si $a, b \in \mathbb{R}$:

$$R^n \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_{n,1} \\ x_{n,2} \end{pmatrix}$$

$$= \frac{b - ai}{2} \left(\frac{9}{10} e^{i\pi/4} \right)^n \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{b + ai}{2} \left(\frac{9}{10} e^{-i\pi/4} \right)^n \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

$$x_{n,1} = \left[\frac{b - ai}{2} \left(\frac{9}{10} e^{i\pi/4} \right)^n - \frac{b + ai}{2} \left(\frac{9}{10} e^{-i\pi/4} \right)^n \right] i$$

puisque $z - \overline{z} = 2i\,\mathfrak{Im}(z)$,

$$x_{n,1} = \left(\frac{9}{10} \right)^n (a \cos(n\pi/4) - b \sin(n\pi/4))$$

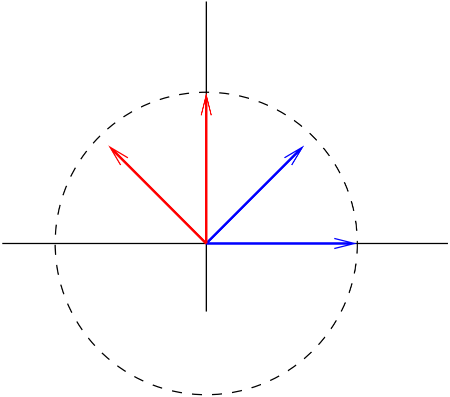
$$x_{n,2} = \frac{b - ai}{2} \left(\frac{9}{10} e^{i\pi/4} \right)^n + \frac{b + ai}{2} \left(\frac{9}{10} e^{-i\pi/4} \right)^n$$

puisque $z + \overline{z} = 2\,\mathfrak{Re}(z)$,

$$x_{n,2} = \left(\frac{9}{10} \right)^n (a \sin(n\pi/4) + b \cos(n\pi/4))$$

On retrouve le fait suivant : si $a, b \in \mathbb{R}$, alors

$$R\begin{pmatrix} a \\ b \end{pmatrix} = aR\begin{pmatrix} 1 \\ 0 \end{pmatrix} + bR\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



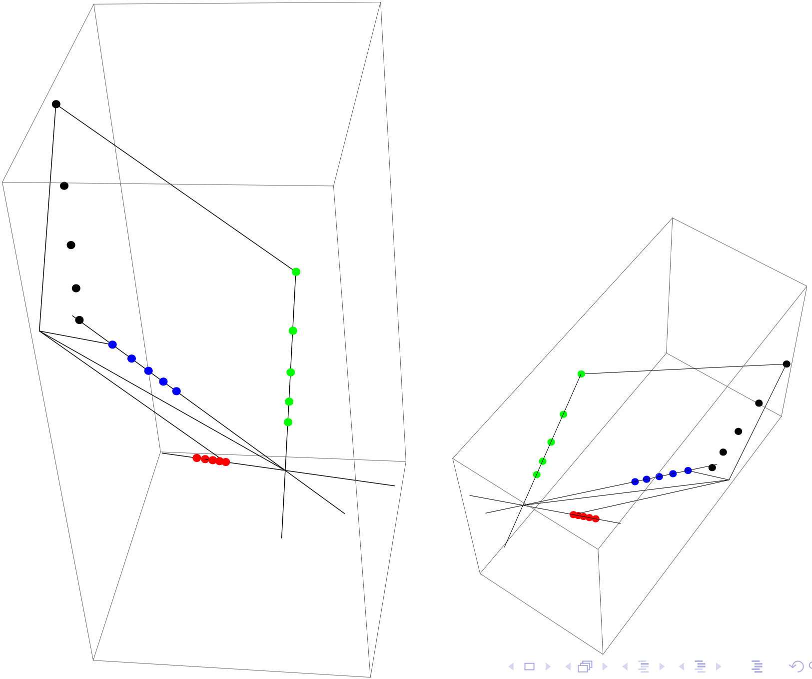
Un dernier exemple dans $\mathbb{R}^3$,

$$S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

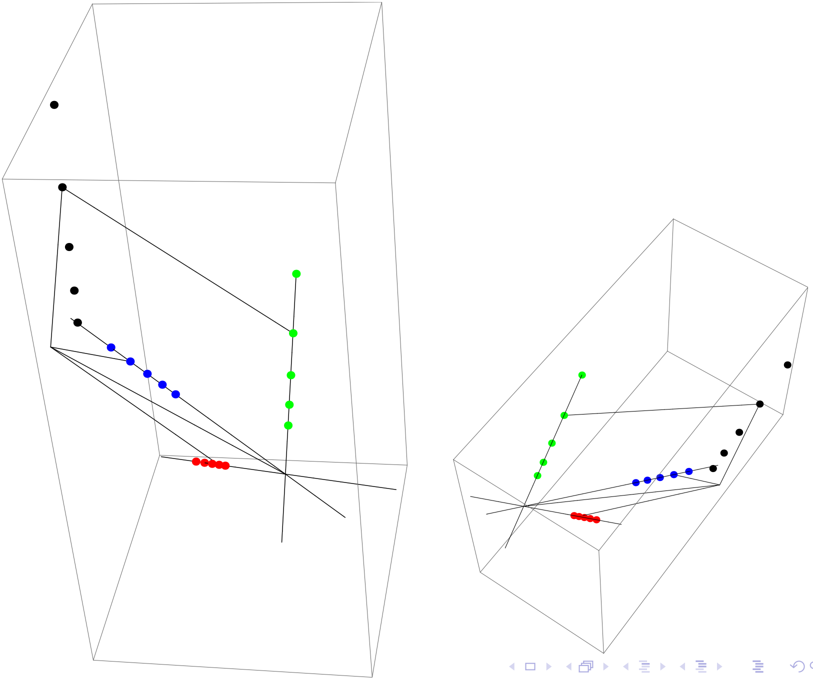
$$A = S \begin{pmatrix} 1.1 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.7 \end{pmatrix} S^{-1} = \frac{1}{10} \begin{pmatrix} 15 & -2 & -2 \\ 8 & 5 & -2 \\ 4 & 0 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 7/2 \\ 1/2 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{5}{2} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

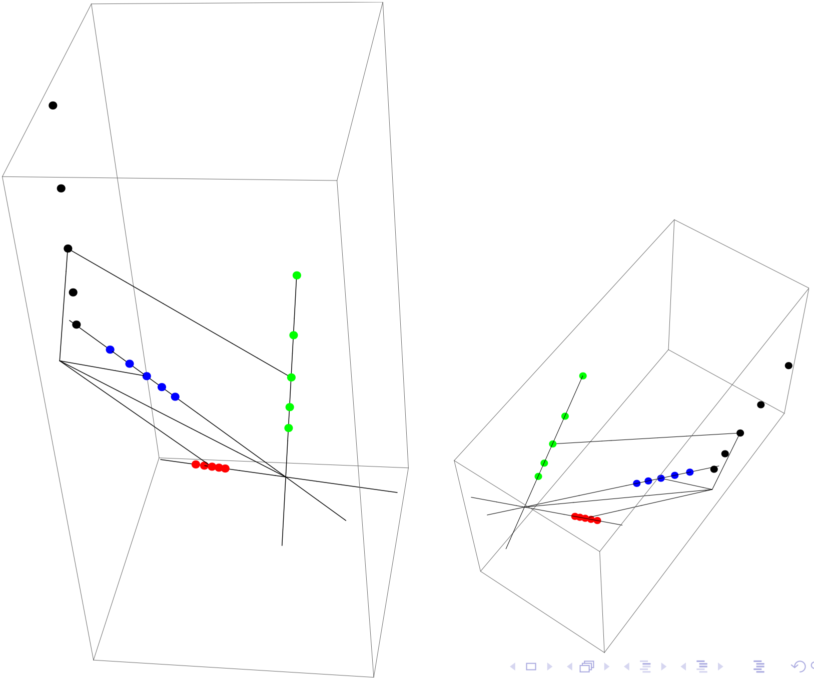
$A^0(7/2, 1/2, 9)^\sim \qquad \mathbb{R}^3 = E_{1.1} \oplus E_{0.9} \oplus E_{0.7}$



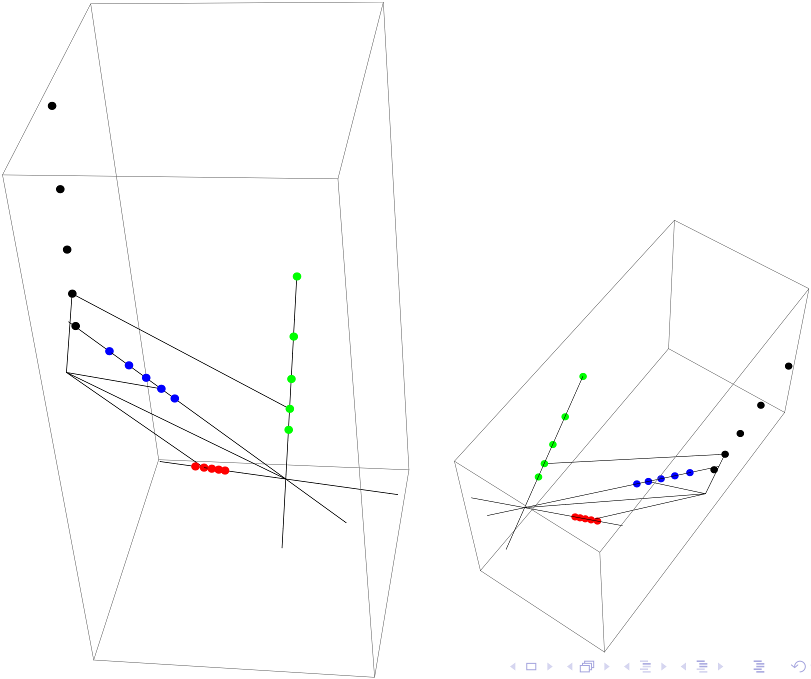
$A^1(7/2, 1/2, 9)^{\sim}$ $\qquad$ $\mathbb{R}^3 = E_{1.1} \oplus E_{0.9} \oplus E_{0.7}$



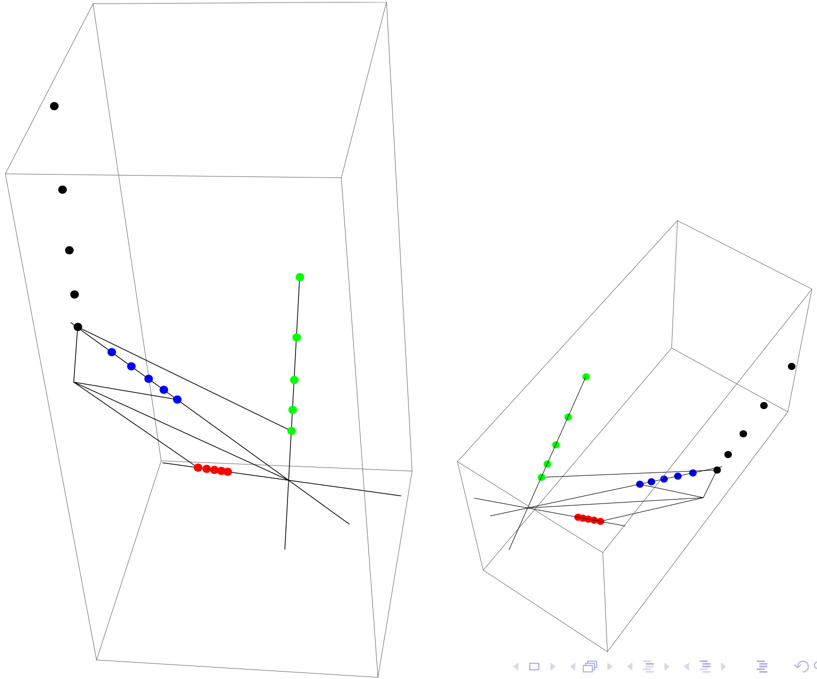
$A^2(7/2, 1/2, 9)^{\sim}$ $\quad \mathbb{R}^3 = E_{1.1} \oplus E_{0.9} \oplus E_{0.7}$



$A^3(7/2, 1/2, 9)^{\sim}$ $\quad \mathbb{R}^3 = E_{1.1} \oplus E_{0.9} \oplus E_{0.7}$



$A^4(7/2, 1/2, 9)^{\sim}$ $\mathbb{R}^3 = E_{1.1} \oplus E_{0.9} \oplus E_{0.7}$



$$A^n = (S\Delta S^{-1})^n = S\,\Delta^n S^{-1} \text{ avec } \Delta = \mathrm{diag}(1.1,\, 0.9,\, 0.7)$$

donc,

$$A^n = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1.1^n & 0 & 0 \\ 0 & 0.9^n & 0 \\ 0 & 0 & 0.7^n \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 \\ -2 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3\,1.1^n - 2\,0.9^n & 0.9^n - 1.1^n & 0.9^n - 1.1^n \\ -0.7^n + 3\,1.1^n - 2\,0.9^n & 0.7^n + 0.9^n - 1.1^n & 0.9^n - 1.1^n \\ 0.7^n + 3\,1.1^n - 4\,0.9^n & -0.7^n - 1.1^n + 2\,0.9^n & -1.1^n + 2\,0.9^n \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{pmatrix}}_{\Delta^n} = \begin{pmatrix} a\,\lambda_1^n & b\,\lambda_2^n & c\,\lambda_3^n \\ d\,\lambda_1^n & e\,\lambda_2^n & f\,\lambda_3^n \\ g\,\lambda_1^n & h\,\lambda_2^n & i\,\lambda_3^n \end{pmatrix}$$

$$S\Delta^n S^{-1} = \begin{pmatrix} a\,\lambda_1^n & b\,\lambda_2^n & c\,\lambda_3^n \\ d\,\lambda_1^n & e\,\lambda_2^n & f\,\lambda_3^n \\ g\,\lambda_1^n & h\,\lambda_2^n & i\,\lambda_3^n \end{pmatrix} \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix}$$

D'une manière générale, si A est une matrice diagonalisable $r \times r$, $A = S\Delta S^{-1}$ où $\Delta = \mathrm{diag}(\lambda_1 \cdots \lambda_r)$, alors

$$[A^n]_{i,j} = \sum_{p=1}^{r} c_p^{(i,j)} \lambda_p^n$$