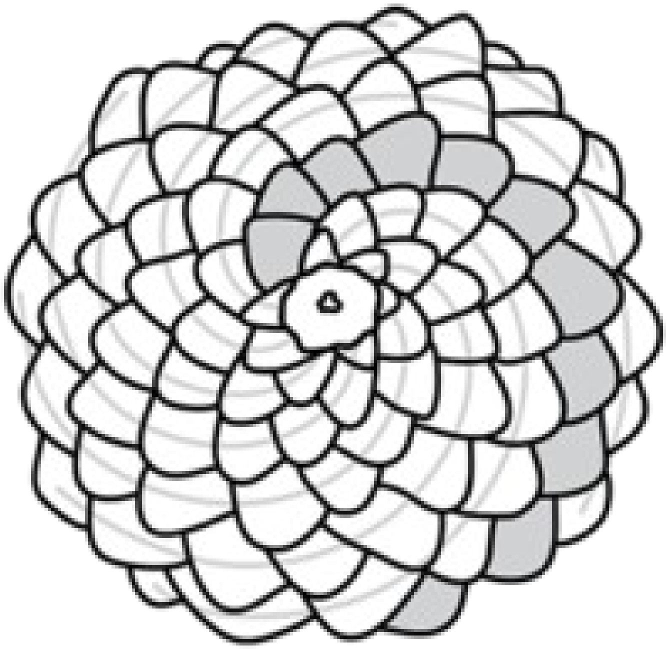
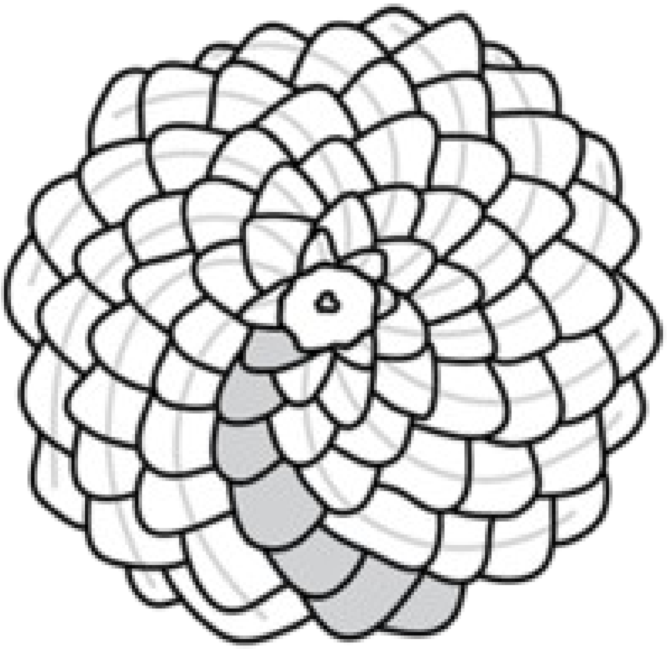
Diagonalisation et fractions simples

M. Rigo, mars 2018

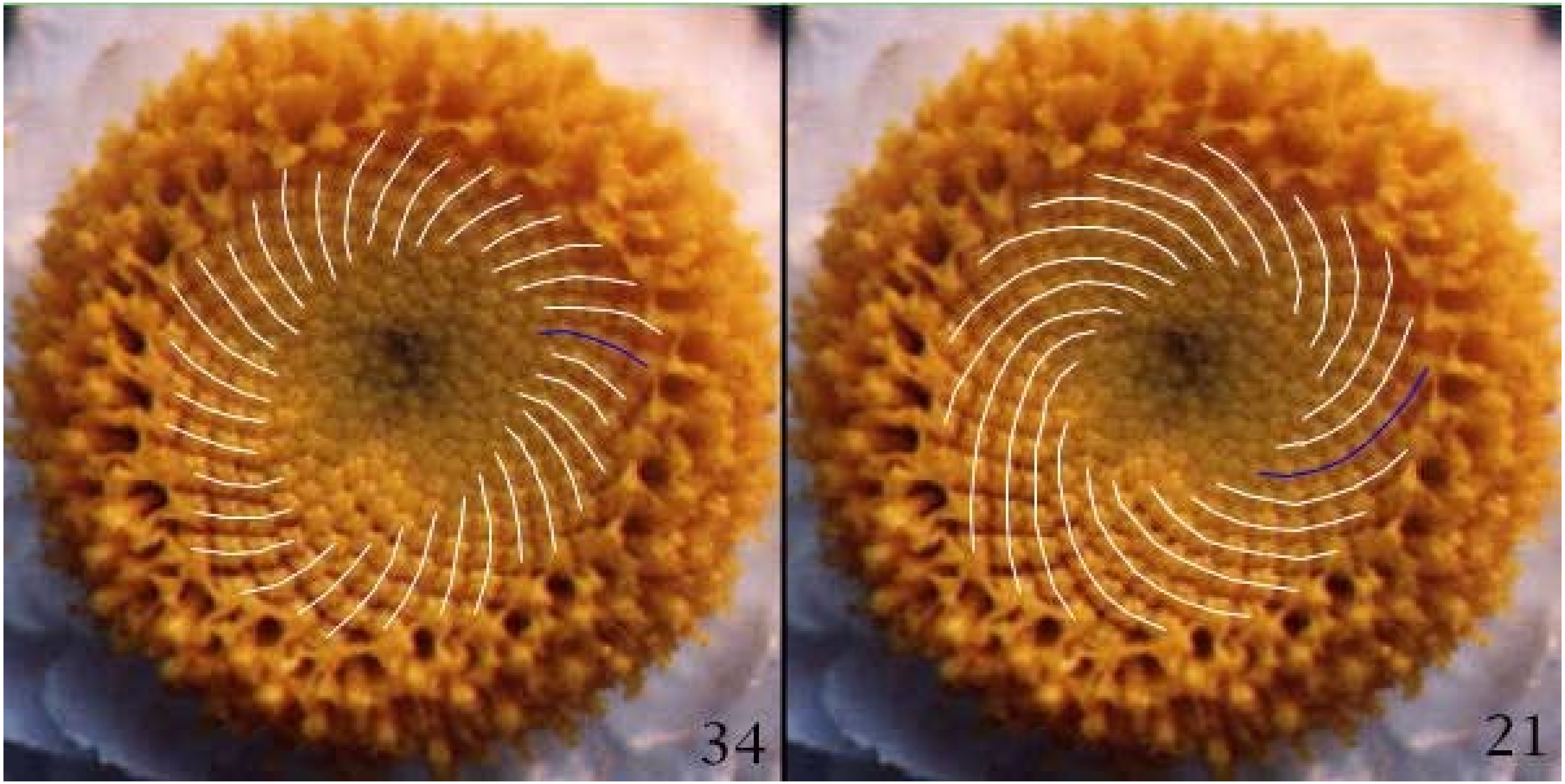
La **phyllotaxie** est l’ordre dans lequel sont implantés les feuilles ou les rameaux sur la tige d’une plante, ou, par extension, la disposition des éléments d’un fruit, d’une fleur, d’un bourgeon. (*Wikipedia*)

La **phyllotaxie** désigne également la science qui étudie ces arrangements.





http://curiosamathematica.tumblr.com/post/114294953296/golden-ratio-in-phyllotaxis



http://scotton.freeshell.org/phyllo/phyllo.html

Nombre de tours / nombre de feuilles avant de retrouver la même configuration

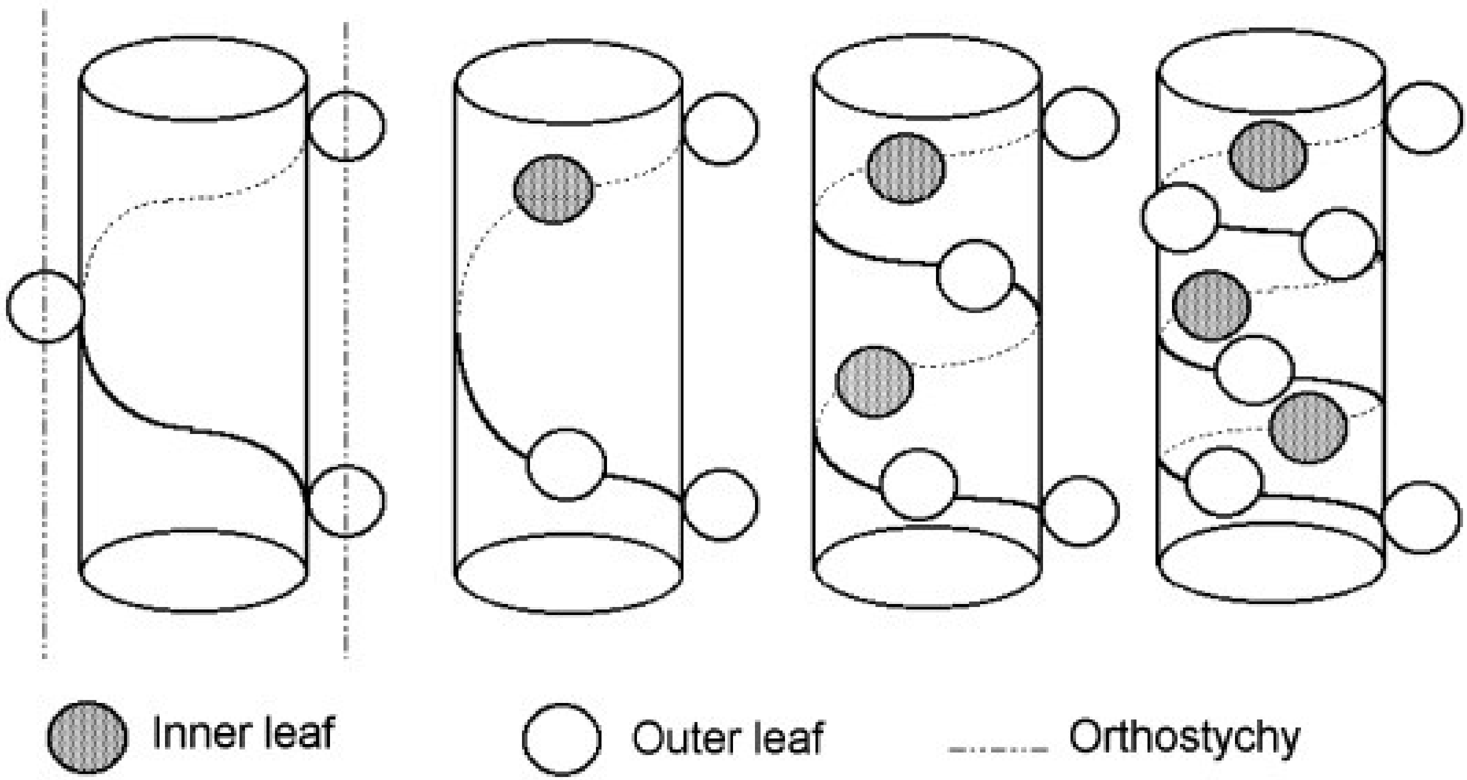


Figure 5.20. Four first Fibonacci types of spiral phyllotaxis: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{8}$.

http://www.brainkart.com/article/Morphology-of-the-Stem_17042/

Soft Condensed Matter, arXiv:1203.6257

Fibonacci numbers in phyllotaxis, Gilbert Zalczer

Service de Physique de l’Etat Condensé, Saclay, Gif-sur-Yvette

[…] very elaborate models, considering that the appearance of a new leaf is conditioned by a repulsive potential created by all the previous ones, writing down explicit forms for the model potential, the growth law and the conditions of appearance and computing or simulating the solutions [...]

S. Douady and Y. Couder, J. Theor. Biol. 178, 255 (1996)

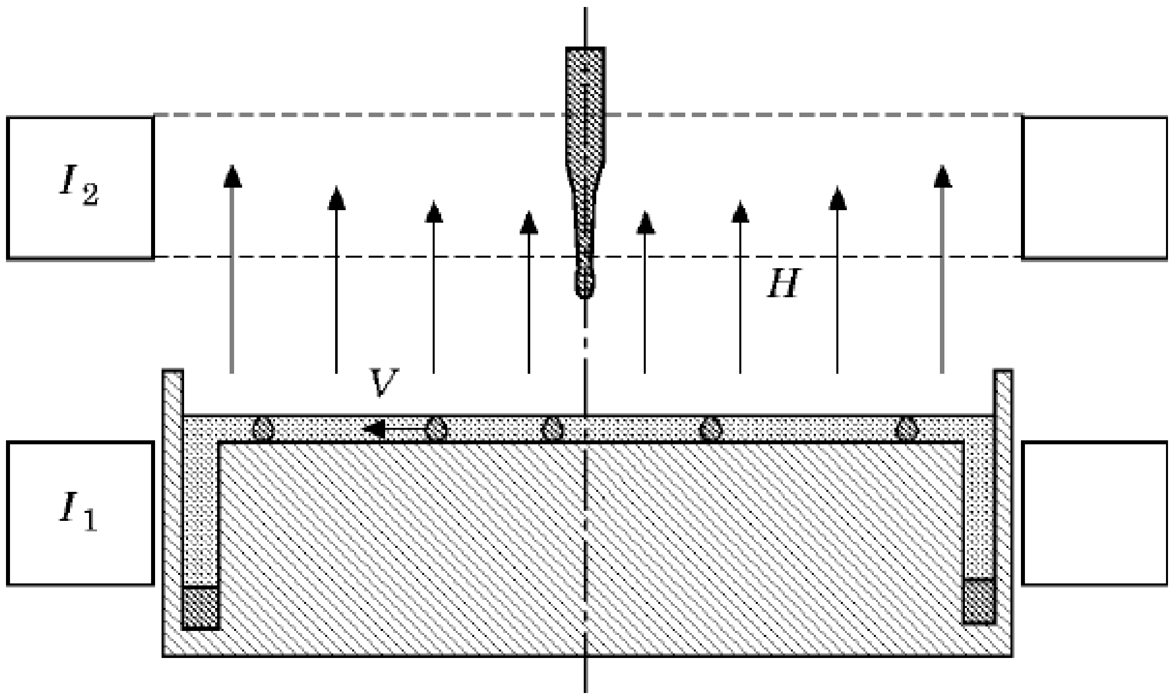
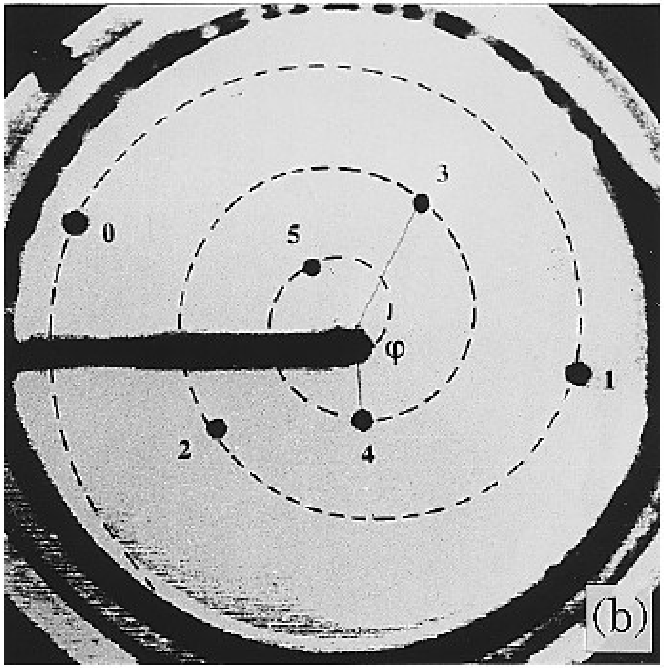
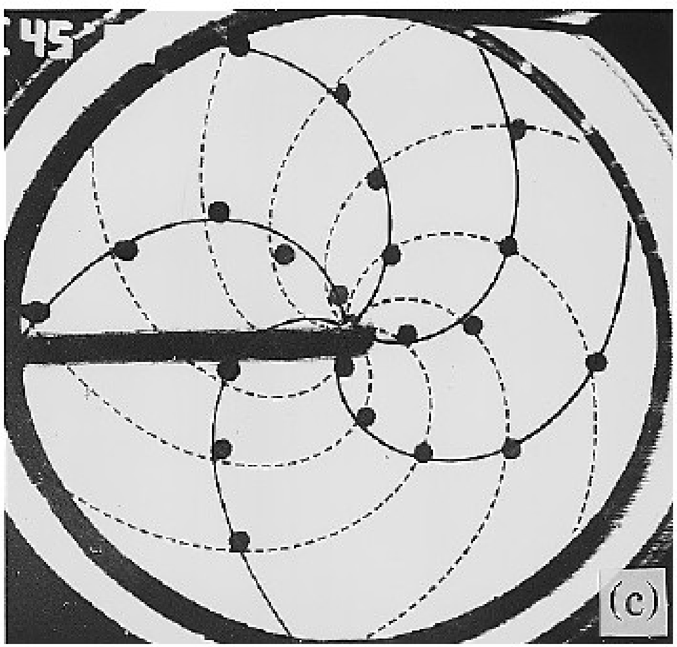


FIG. 2. Sketch of the experimental apparatus. Drops of ferrofluid are used to simulate the primordia. The drops (of volume v110 mm3) fall with a tunable periodicity T at the centre of a horizontal teflon dish. The vertical magnetic field H is created by two coils in the Helmholtz position. The dipoles are radially advected with velocity V by the magnetic field gradient (controlled by the currents I1 and I2 in the two coils). The drops ultimately fall into a deep ditch at the periphery, designed to prevent accumulation.





Typical phyllotactic patterns formed by the ferrofluid drops for different values of the control parameter G.

The drops are visible as dark dots. The tube for the ferrofluid supply partially hides the central truncated cone. The drops are numbered in their order of formation.

Doodling in Math: Spirals, Fibonacci, and Being a Plant [1 of 3]

https://youtu.be/ahXIMUkSXX0

Suite de Fibonacci (Leonardo de Pise $\simeq$ 1175–1250)

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ \ldots$$

$$F_{n+2} = F_{n+1} + F_n \quad \forall n \geq 0$$

$$F_0 = F_1 = 1$$

Peut-on trouver une formule "close" pour F_n ?

Matrice *compagnon*

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Exprimer la puissance n-ième d'une matrice. . .

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Exprimer la puissance n-ième d'une matrice...

$$\chi(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 1$$

2 valeurs propres simples :

$$\varphi = \frac{1}{2}\left(1+\sqrt{5}\right) \quad \text{et} \quad \varphi' = \quad \frac{1}{2}\left(1-\sqrt{5}\right)$$

$$S = \begin{pmatrix} \varphi & \varphi' \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} S = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi' \end{pmatrix}$$

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$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = S\begin{pmatrix}\varphi^n & 0\\ 0 & \varphi'^n\end{pmatrix} S^{-1}$$

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n = \begin{pmatrix}\frac{\varphi^{n+1}-\varphi'^{n+1}}{\sqrt{5}} & \frac{\varphi^n-\varphi'^n}{\sqrt{5}}\\ \frac{\varphi^n-\varphi'^n}{\sqrt{5}} & \frac{\varphi\,\varphi'^n-\varphi'\,\varphi^n}{\sqrt{5}}\end{pmatrix}$$

On retrouve le fait que *les éléments de M^n s'expriment comme combinaisons linéaires de puissances n-ièmes des valeurs propres de M.*

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$\forall n \geq 0$,

$$F_n = \frac{\varphi^n - \varphi'^n}{\sqrt{5}} + \frac{\varphi\,\varphi'^n - \varphi'\,\varphi^n}{\sqrt{5}}$$

$$F_n = \frac{(1-\varphi')\,\varphi^n - (1-\varphi)\,\varphi'^n}{\sqrt{5}}$$

$$F_n = \frac{\sqrt{5}}{5}\left(\varphi^{n+1} - \varphi'^{n+1}\right)$$

connu parfois comme formule de Binet (1786–1856)

$$\lim_{n\to+\infty} \frac{F_n}{\frac{\sqrt{5}}{5}\varphi^{n+1}} = 1$$

car

$$\varphi \simeq 1,618 \quad \text{et} \quad \varphi' \simeq -0,618, \qquad \lim_{n\to+\infty} \varphi'^n = 0.$$

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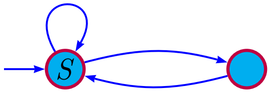
Nombre de chemins de longueur n partant de et arrivant en S



Nombre de mots sur $\{a, b\}$ sans bb

Une série "formelle" code une suite :

$$\mathsf{s}_F(z) = \sum_{n=0}^{+\infty} F_n\, z^n = 1 + 1.z + 2.z^2 + 3.z^3 + 5.z^4 + 8.z^5 + \cdots$$

Théorème

La série s_F est une fraction rationnelle SSI $(F_n)_{n\geq 0}$ satisfait une relation de récurrence linéaire.

$$\mathsf{s}_F(z) = \frac{1}{1 - z - z^2} = \frac{1}{z^2 \chi(1/z)}$$

Décomposition en fractions simples

$$\frac{1}{1-z-z^2} = \frac{\sqrt{5}}{5(z+\varphi)} - \frac{\sqrt{5}}{5(z+\varphi')}$$

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n$$

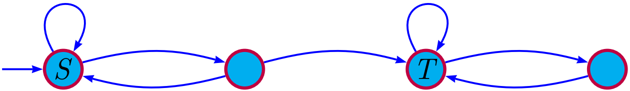
$$\frac{1}{z+\varphi} = -\frac{\varphi'}{1-\varphi'\,z} = -\sum_{n=0}^{+\infty} \varphi'^{n+1}\, z^n$$

$$\frac{1}{z+\varphi'} = -\frac{\varphi}{1-\varphi\,z} = -\sum_{n=0}^{+\infty} \varphi^{n+1}\, z^n$$

$$G_{n+4} = 2G_{n+3} + G_{n+2} - 2G_{n+1} - G_n$$

$$\begin{pmatrix} G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} G_3 \\ G_2 \\ G_1 \\ G_0 \end{pmatrix}$$

Nombre de chemins de longueur n partant de S et arrivant en T



$$M = \begin{pmatrix} 2 & 1 & -2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ n'est pas diagonalisable!}$$

$$S = \begin{pmatrix} 2-\sqrt{5} & -\frac{3}{2}\left(-3+\sqrt{5}\right) & 2+\sqrt{5} & \frac{3}{2}\left(3+\sqrt{5}\right) \\ \frac{1}{2}\left(3-\sqrt{5}\right) & 1-\sqrt{5} & \frac{1}{2}\left(3+\sqrt{5}\right) & 1+\sqrt{5} \\ \frac{1}{2}\left(1-\sqrt{5}\right) & 1 & \frac{1}{2}\left(1+\sqrt{5}\right) & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Forme de Jordan

$$S^{-1}MS = \begin{pmatrix} \varphi' & 1 & 0 & 0 \\ 0 & \varphi' & 0 & 0 \\ 0 & 0 & \varphi & 1 \\ 0 & 0 & 0 & \varphi \end{pmatrix}$$

Puissance n-ième d'un bloc de Jordan

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\,\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\,\lambda^{n-1} & \frac{(n-1)n}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\,\lambda^{n-1}n \\ 0 & 0 & \lambda^n \end{pmatrix}$$