Structural Properties of bounded Languages with Respect to Multiplication by a Constant

Emilie Charlier\textsuperscript{1}  Michel Rigo\textsuperscript{1}  Wolfgang Steiner\textsuperscript{2}

\textsuperscript{1}Department of Mathematics  
University of Liège

\textsuperscript{2}University Paris 7 / LIAFA / CNRS

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Outline of the talk

Abstract Numeration Systems

Motivation - Main Question

First Results

Bounded Languages

$B_{\ell}$-Representation of an Integer

Multiplication by $\lambda = \beta^\ell$
Abstract Numeration Systems

Definition (P. Lecomte, M. Rigo)

An abstract numeration system is a triple $S = (L, \Sigma, <)$ where $L$ is a regular language over a totally ordered alphabet $(\Sigma, <)$. Enumerating the words of $L$ with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$
\text{rep}_S : \mathbb{N} \rightarrow L \quad \text{val}_S = \text{rep}_S^{-1} : L \rightarrow \mathbb{N}.
$$

Example

$L = a^*$, $\Sigma = \{a\}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>rep($n$)</td>
<td>$\varepsilon$</td>
<td>$a$</td>
<td>$aa$</td>
<td>$aaa$</td>
<td>$aaaa$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>
Abstract Numeration Systems

Example

$L = \{a, b\}^*, \Sigma = \{a, b\}, \ a < b$

<table>
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<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>⋯</th>
</tr>
</thead>
<tbody>
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<td>$\varepsilon$</td>
<td>$a$</td>
<td>$b$</td>
<td>$aa$</td>
<td>$ab$</td>
<td>$ba$</td>
<td>$bb$</td>
<td>$aaa$</td>
<td>⋯</td>
</tr>
</tbody>
</table>

Example

$L = a^* b^*, \Sigma = \{a, b\}, \ a < b$

<table>
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<tr>
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val($a^p b^q$) = $\frac{1}{2}(p + q)(p + q + 1) + q$
Abstract Numeration Systems

#b

#a
Abstract Numeration Systems
Abstract Numeration Systems
Abstract Numeration Systems
Abstract Numeration Systems
Abstract Numeration Systems
Abstract Numeration Systems

#b

#a
Abstract Numeration Systems

#b

#a
Abstract Numeration Systems

Remark
This generalizes “classical” Pisot systems like integer base systems or Fibonacci system.

\[ L = \{ \varepsilon \} \cup \{ 1, \ldots, k - 1 \}\{0, \ldots, k - 1 \}^* \] or \[ L = \{ \varepsilon \} \cup 1\{0, 01\}^* \]

Definition
A set \( X \subseteq \mathbb{N} \) is \( S \)-recognizable if \( \text{rep}_S(X) \subseteq \Sigma^* \) is a regular language (accepted by a DFA).
Motivation - Main Question

How to compute in such a numeration system? More precisely, how act arithmetic operations like addition, multiplication by a constant, . . .?

We focus on multiplication by a constant.

Question: Multiplication by a Constant

If \( S = (L, \Sigma, <) \) is an abstract numeration system, can we find some necessary and sufficient condition on \( \lambda \in \mathbb{N} \) such that for any \( S \)-recognizable set \( X \), the set \( \lambda X \) is still \( S \)-recognizable?

\[ X \text{ S-rec} \quad \Rightarrow \quad \lambda X \text{ S-rec} \]
First Results

Theorem (Translation, P. Lecomte, M. Rigo)
Let $S = (L, \Sigma, \prec)$ be an abstract numeration system and $X \subseteq \mathbb{N}$. For each $t \in \mathbb{N}$, $X + t$ is $S$-recognizable if and only if $X$ is $S$-recognizable.

Definition
We denote by $u_L(n)$ the number of words of length $n$ belonging to $L$.

Theorem (Polynomial Case, M. Rigo)
Let $L \subseteq \Sigma^*$ be a regular language such that $u_L(n)$ is $\Theta(n^k)$ for some $k \in \mathbb{N}$ and $S = (L, \Sigma, \prec)$. Preservation of $S$-recognizability after multiplication by $\lambda$ holds only if $\lambda = \beta^{k+1}$ for some $\beta \in \mathbb{N}$.
First Results

Definition
A language $L$ is slender if $u_L(n) \in O(1)$.

Theorem (Slender Case, E. C., M. Rigo)
Let $L \subset \Sigma^*$ be a slender regular language and $S = (L, \Sigma, <)$. A set $X \subseteq \mathbb{N}$ is $S$-recognizable if and only if $X$ is a finite union of arithmetic progressions.

Corollary
Let $S$ be a numeration system built on a slender language. If $X \subseteq \mathbb{N}$ is $S$-recognizable then $\lambda X$ is $S$-recognizable for all $\lambda \in \mathbb{N}$.
First Results

Theorem (P. Lecomte, M. Rigo)

Let $\beta \in \mathbb{N} \setminus \{0\}$. For the abstract numeration system

$$S = (a^* b^*, \{a, b\}, a < b),$$

multiplication by $\beta^2$ preserves $S$-recognizability if and only if $\beta$ is an odd integer.

$\rightarrow$ We focus on abstract numeration systems built on bounded languages.
Bounded Languages

Notation
We denote by $B_\ell = a_1^* \cdots a_\ell^*$ the bounded language over the totally ordered alphabet $\Sigma_\ell = \{a_1 < \ldots < a_\ell\}$ of size $\ell \geq 1$.

We consider abstract numeration systems of the form $(B_\ell, \Sigma_\ell)$ and we denote by $\text{rep}_\ell$ and $\text{val}_\ell$ the corresponding bijections.

A set $X \subseteq \mathbb{N}$ is said to be $B_\ell$-recognizable if $\text{rep}_\ell(X)$ is a regular language over the alphabet $\Sigma_\ell$.

If $w$ is a word over $\Sigma_\ell$, $|w|$ denotes its length and $|w|_{a_j}$ counts the number of letters $a_j$’s appearing in $w$. The Parikh mapping $\Psi$ maps a word $w \in \Sigma_\ell^*$ onto the vector $\Psi(w) := (|w|_{a_1}, \ldots, |w|_{a_\ell})$. 
Bounded Languages

In this context, multiplication by a constant \( \lambda \) can be viewed as a transformation

\[ f_\lambda : B_\ell \to B_\ell. \]

The question becomes then:

*Can we determine some necessary and sufficient condition under which this transformation preserves regular subsets of \( B_\ell \)?*

**Example**

Let \( \ell = 2, \Sigma_2 = \{a, b\} \) and \( \lambda = 25 \).

Thus multiplication by \( \lambda = 25 \) induces a mapping \( f_\lambda \) onto \( B_2 \) such that for \( w, w' \in B_2 \), \( f_\lambda(w) = w' \) if and only if \( \text{val}_2(w') = 25 \text{val}_2(w) \).
$\mathcal{B}_\ell$-Representation of an Integer

We set

$$u_\ell(n) := u_{B_\ell}(n) = \#(B_\ell \cap \Sigma^n) \quad \text{and} \quad v_\ell(n) := \#(B_\ell \cap \Sigma^{\leq n}) = \sum_{i=0}^{n} u_\ell(i).$$

Lemma

For all integers $\ell \geq 1$ and $n \geq 0$, we have

$$u_{\ell+1}(n) = v_\ell(n) \quad \text{and} \quad u_\ell(n) = \binom{n + \ell - 1}{\ell - 1}.$$
Lemma

Let $\ell \in \mathbb{N} \setminus \{0\}$ and $S = (a_1^* \cdots a_\ell^*, \{a_1 < \cdots < a_\ell\})$. We have

$$\text{val}_\ell(a_1^{n_1} \cdots a_\ell^{n_\ell}) = \sum_{i=1}^{\ell} \left( \binom{n_i + \cdots + n_\ell + \ell - i}{\ell - i + 1} \right).$$

Corollary (Lehmer 1964, Katona 1966, Fraenkel 1982)

Let $\ell \in \mathbb{N} \setminus \{0\}$. Any positive integer $n$ can be uniquely written as

$$n = \binom{z_\ell}{\ell} + \binom{z_{\ell-1}}{\ell - 1} + \cdots + \binom{z_1}{1}$$

with $z_\ell > z_{\ell-1} > \cdots > z_1 \geq 0$. 
Example
Consider the words of length 3 in the language $a^* b^* c^*$,

$$aaa < aab < aac < abb < abc < acc < bbb < bbc < bcc < ccc.$$

We have $\text{val}_3(aaa) = \binom{5}{3} = 10$ and $\text{val}_3(acc) = 15$. If we apply the erasing morphism $\varphi : \{a, b, c\} \to \{a, b, c\}^*$ defined by

$$\varphi(a) = \varepsilon, \varphi(b) = b, \varphi(c) = c$$

on the words of length 3, we get

$$\varepsilon < b < c < bb < bc < cc < bbb < bbc < bcc < ccc.$$

So we have $\text{val}_3(acc) = \text{val}_3(aaa) + \text{val}_2(cc)$ where $\text{val}_2$ is considered as a map defined on the language $b^* c^*$. 
Algorithm computing $\text{rep}_\ell(n)$.

Let $n$ be an integer and $\ell$ be a positive integer. For $i=\ell, \ell-1, \ldots, 1$ do

- if $n>0$,
  - find $t$ such that $\binom{t}{i} \leq n < \binom{t+1}{i}$
  - $z(i) \leftarrow t$
  - $n \leftarrow n - \binom{t}{i}$
- otherwise, $z(i) \leftarrow i-1$

Consider now the triangular system having $\alpha_1, \ldots, \alpha_\ell$ as unknowns

$$\alpha_i + \cdots + \alpha_\ell = z(\ell - i + 1) - \ell + i, \quad i = 1, \ldots, \ell.$$ 

One has $\text{rep}_\ell(n) = a_1^{\alpha_1} \cdots a_\ell^{\alpha_\ell}$. 
Multiplication by $\lambda = \beta^\ell$

Remark
We have $u_{B_\ell}(n) \in \Theta(n^{\ell-1})$.
So we have to focus only on multiplicators of the kind

$$\lambda = \beta^\ell.$$  

Lemma
Let $\ell, \beta \in \mathbb{N} \setminus \{0\}$. For $n \in \mathbb{N}$ large enough, we have

$$|\text{rep}_\ell(\beta^\ell n)| = \beta |\text{rep}_\ell(n)| + \left[ \frac{(\beta - 1)(\ell + 1)}{2} \right] - i$$

with $i \in \{0, 1, \ldots, \beta\}$. 

**Lemma**

Let $\ell, \beta \in \mathbb{N} \setminus \{0\}$. Define $c_\ell, c_{\ell-1}, \ldots, c_1$ recursively by

$$c_{k+1} = k! (\beta^{\ell-k} - 1) \sum_{i=k}^\ell \frac{S_1(i, k)}{i!} - \sum_{i=k+2}^\ell \sum_{j=k+1}^i \frac{S_1(i, j) j!}{i! (j-k)!} c_{j-k}$$

where $S_1(i, j)$ are the unsigned Stirling numbers of the first kind.

Then we have

$$\beta^\ell \left( \binom{q + \ell}{\ell} + \binom{q + \ell - 1}{\ell - 1} + \cdots + \binom{q}{1} \right) = \binom{\beta q + c_\ell + \ell - 1}{\ell} + \binom{\beta q + c_{\ell-1} + \ell - 2}{\ell - 1} + \cdots + \binom{\beta q + c_1}{1},$$

for all $q \in \mathbb{R}$.
Multiplication by $\lambda = \beta^\ell$

**Remark**

If all $c_k$, $1 \leq k \leq \ell$, are integers and $c_\ell \geq c_{\ell-1} \geq \cdots \geq c_1$, then

$$\text{rep}_\ell(\beta^\ell \text{val}_\ell(a_q^\ell)) = a_1^{c_\ell-c_{\ell-1}} a_2^{c_{\ell-1}-c_{\ell-2}} \cdots a_{\ell-1}^{c_2-c_1} a_{\ell}^{\beta q+c_1}$$

for all $q \geq -c_1/\beta$, hence $f_{\beta^\ell}(a_n^*)$ is regular.

Explicit forms for $c_\ell$ and $c_{\ell-1}$:

$$c_\ell = \frac{\beta - 1)(\ell + 1)}{2} \quad \text{for } \ell \geq 2,$$

$$c_{\ell-1} = \frac{\beta - 1)(\ell + 1)}{2} - \frac{\beta^2 - 1)(\ell + 1)}{24} \quad \text{for } \ell \geq 3.$$
Multiplication by $\lambda = \beta^\ell$

**Lemma**

Let $A$ be a $k$-dimensional linear subset of $\mathbb{N}^\ell$ for some integer $1 \leq k < \ell$ and $B = \Psi^{-1}(A) \cap B_\ell$ be the corresponding subset of $B_\ell$. If $\Psi(f_{\beta^\ell}(B))$ contains a sequence $x^{(n)} = (x_1^{(n)}, \ldots, x_\ell^{(n)})$ such that $
abla \min(x_{j_1}^{(n)}, \ldots, x_{j_{k+1}}^{(n)}) \to \infty$ as $n \to \infty$ for some $j_1 < \cdots < j_{k+1}$, then $f_{\beta^\ell}(B)$ is not regular.

**Proposition**

If $c_\ell \not\in \mathbb{Z}$ or $c_{\ell-1} \not\in \mathbb{Z}$ with $\ell \geq 3$, then $f_{\beta^\ell}(a_\ell^*)$ is not regular.

**Proposition**

If $c_\ell, c_{\ell-1} \in \mathbb{Z}$ with $\ell \geq 3$, $\beta \geq 2$, then $f_{\beta^\ell}(a_1^*a_\ell^*)$ is not regular.
Theorem (E. C., M. Rigo, W. Steiner)

Let $\ell, \beta \in \mathbb{N} \setminus \{0\}$. For the abstract numeration system

$$S = (a_1^* \ldots a_\ell^*, \{a_1 < \ldots < a_\ell\}),$$

multiplication by $\beta^\ell$ preserves $S$-recognizability if and only if one of the following condition is satisfied:

- $\ell = 1$
- $\beta = 1$
- $\ell = 2$ and $\beta$ is an odd integer.