# Positional Numeration Systems Without a Dominant Root: Emergence of Alternate Real Bases

Émilie Charlier

Département de mathématique, Université de Liège, Belgique

Dancing Numbers Symposium, in honor of Karma Dajani, Utrecht
29 August 2025

# Positional numeration systems

Let $U = (U_n)_{n \geq 0}$ be a base sequence, that is, an increasing sequence of integers such that $U_0 = 1$ and the quotients $\frac{U_n}{U_{n-1}}$ are bounded.

A natural number $x$ is represented by the finite word

$$\mathrm{rep}_U(x) = a_{\ell-1} \cdots a_0$$

obtained from the greedy algorithm:

$$x = \sum_{n=0}^{\ell-1} a_n U_n.$$

A description of the numeration language

$$L_U = 0^* \{\mathrm{rep}_U(x) : x \in \mathbb{N}\}$$

strongly depends on the base sequence $U$.

# Regularity of $L_U$

A fundamental question is the following:

- Given a positional system $U$, can we decide if the numeration language $L_U$ is regular?
- And even more precisely, can we characterize those systems $U$ giving rise to a regular numeration language $L_U$?

# Linear systems

A necessary condition is that the sequence $U = (U_n)_{n\geq 0}$ is linear, i.e., it must satisfy a linear recurrence relation with integer coefficients: there exist integers $c_1, \ldots, c_k$ such that

$$U_n = c_1 U_{n-1} + c_2 U_{n-2} \cdots + c_k U_{n-k}, \quad \text{for all } n \geq k.$$

A way to see this is:

- Note that, for each $n$, $U_n$ is the number of words of length $n$ in $L_U$.
- This implies that the formal series
$$S = \sum_{n\geq 0} U_n X^n$$
is $\mathbb{Z}$-rational, i.e., $S = \frac{P}{Q}$ for polynomials $P, Q \in \mathbb{Z}[X]$ with $Q(0) = 1$.
- This, in turn, implies that the sequence $(U_n)_{n\geq 0}$ satisfies a linear recurrence relation over $\mathbb{Z}$.

# Hollander's study for dominant root systems

This question was studied by Hollander in 1998 in the case of positional systems $U = (U_n)_{n\geq 0}$ with a dominant root, i.e., such that the limit $\lim_{n\to\infty} \frac{U_n}{U_{n-1}}$ exists.

A clever observation he made was that it is sufficient to study the regularity of the language made of words of maximal length.

### Proposition (Hollander 1998)

*$L_U$ is regular $\iff$ $\mathrm{Max}(L_U) := \{\mathrm{rep}_U(U_n - 1) : n \geq 0\}$ is regular.*

He also showed the following necessary condition:

### Proposition (Hollander 1998)

*If $U$ has a dominant root $\beta > 1$ and if $L_U$ is regular, then $\beta$ is a Parry number.*

# Hollander's key observation

## Proposition (Hollander 1998)

*If $U$ has a dominant root $\beta > 1$ and if $L_U$ is regular, then $\beta$ is a Parry number.*

Here, the key observation is:

- If $d_\beta(1)$ is infinite, then
$$\lim_{n\to\infty} \mathrm{rep}_U(U_n - 1) = d_\beta(1).$$
- If $d_\beta(1) = d_1 \cdots d_\ell 0^\omega$ with $d_\ell \neq 0$, then for all lengths $L$ and all large enough indices $n$, there exists $j \geq 0$ such that
$$\mathrm{Pref}_L(\mathrm{rep}_U(U_n - 1)) = \mathrm{Pref}_L(\mathbf{w}_j)$$
where $\mathbf{w}_j = (d_1 \cdots d_{\ell-1}(d_\ell - 1))^j d_1 \cdots d_\ell 0^\omega$.

We will refer to these words $\mathbf{w}_j$ as the "intermediate" $\beta$-representations of 1.

# Examples and comments

- Integer bases, Fibonacci, or more generally Bertrand numeration systems.

  Let $U = (1, 2, 3, 5, 8, 13, \dots)$ be the Fibonacci sequence starting with $1, 2$. This system is usually called the Zeckendorf system.

  We have

| $U_n - 1$ | 0 | 1 | 2 | 4 | 7 | 12 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $\mathrm{rep}_U(U_n - 1)$ | $\varepsilon$ | 1 | 10 | 101 | 1010 | 10101 | $\cdots$ |

  We also know that $d^*_\varphi(1) = (10)^\omega$.

- The canonical Bertrand system $U$ associated with $\beta$ is characterized by the property

$$\mathrm{rep}_U(U_n - 1) = \mathrm{Pref}_n(d^*_\beta(1)) \quad \text{for all } n \geq 0.$$

  In this case, the numeration system has the dominant root $\beta$, and the numeration language is regular if and only if $\beta$ is a Parry number.

- A non-canonical Bertrand system.

  Let $U = (U_n)_{n \geq 0}$ starting with $1, 2$ and such that $U_{n+2} = U_{n+1} + U_n + 1$ for $n \geq 0$. We get the sequence $U = (1, 2, 4, 7, 12, 20, 33, 54, \ldots)$ and

| $U_n - 1$ | 0 | 1 | 3 | 6 | 11 | 19 | 32 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $\mathrm{rep}_U(U_n - 1)$ | $\varepsilon$ | 1 | 11 | 110 | 1100 | 11000 | 110000 | $\cdots$ |

- The non-canonical Bertrand system $U$ associated with $\beta$ is characterized by the property

$$\mathrm{rep}_U(U_n - 1) = \mathrm{Pref}_n(d_\beta(1)) \quad \text{for all } n \geq 0.$$

  Again in this case, the numeration system has the dominant root $\beta$, and the numeration language is regular if and only if $\beta$ is a Parry number.

- Now, consider the first intermediate $\varphi$-representation $\mathbf{w}_1 = 10110^\omega$ of 1.

Define $U = (U_n)_{n \geq 0}$ in the Bertrand fashion:

$$\begin{aligned} U_0 &= 1, \\ U_1 &= U_0 + 1 = 2, \\ U_2 &= U_1 + 1 = 3, \\ U_3 &= U_2 + U_0 + 1 = 5, \\ U_n &= U_{n-1} + U_{n-3} + U_{n-4} + 1, \quad n \geq 4. \end{aligned}$$

We get $U = (1, 2, 3, 5, 9, 15, 24, 39, 64 \ldots)$ and

| $U_n - 1$ | 0 | 1 | 2 | 4 | 8 | 14 | 23 | 38 | 63 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{rep}_U(U_n - 1)$ | $\varepsilon$ | 1 | 10 | 101 | 1100 | 11000 | 101100 | 1011000 | 11000000 | $\cdots$ |

or, more formally,

$$\mathrm{rep}_U(U_n) = \begin{cases} \mathrm{Pref}_n(10110^\omega), & \text{if } n \equiv 2, 3 \pmod 4; \\ \mathrm{Pref}_n(110^\omega), & \text{if } n \equiv 0, 1 \pmod 4. \end{cases}$$

We see that the limit $\lim_{n\to\infty} \mathrm{rep}_U(U_n - 1)$ does not exist. Nevertheless, the system has a dominant root and the numeration language is regular.

- Consider $U = (U_n)_{n \geq 0}$ defined by $U_n = n3^n + 1$.
  We get $U = (1, 4, 19, 82, 325, 1216, 4375, 15310, 52489, 177148, \ldots)$ and

| $U_n - 1$ | $\text{rep}_U(U_n - 1)$ | $U_n - 1$ | $\text{rep}_U(U_n - 1)$ |
|---|---|---|---|
| 0 | $\varepsilon$ | 15309 | 3123333 |
| 3 | 3 | 52488 | 31123332 |
| 18 | 42 | 177147 | 310320333 |
| 81 | 411 | 590490 | 3101123331 |
| 324 | 3402 | 1948617 | 30310320330 |
| 1215 | 32400 | 6377292 | 302310320322 |
| 4374 | 320400 | 20726199 | 3022101123321 |

  This system is linear and has the dominant root 3. It even satisfies

$$\lim_{n \to \infty} \text{rep}_U(U_n - 1) = 30^\omega$$

  although the convergence speed is very slow. We see a second 0 for $n \geq 30$ and a third one only for $n \geq 85$:

$$\text{rep}_U(U_{30} - 1) = 300301010202112212211202320221$$

$$\text{rep}_U(U_{85} - 1) = 3000300100111012211201300110102200100202211020002021012022210301022201020022101123000.$$

  However, because of the growing suffixes, the numeration language is not regular.

- Consider $U = (U_n)_{n\ge 0}$ starting with $1, 2, 3, 7$ and such that $U_n = 3U_{n-2} + U_{n-4}$ for $n \ge 4$.

  We get $U = (1, 2, 3, 7, 10, 23, 33, 76, \ldots)$.

  This system has no dominant root.

  However, the numeration language is regular:

| $U_n - 1$ | 0 | 1 | 2 | 6 | 9 | 22 | 32 | 75 | 108 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{rep}_U(U_n - 1)$ | $\varepsilon$ | 1 | 10 | 200 | 1010 | 20010 | 101010 | 2001010 | 10101010 | $\cdots$ |

  These maximal words are the prefixes of the two ultimately periodic infinite words $(10)^\omega$ and $20(01)^\omega$.

  These infinite words are not random. To understand them, we have to represent 1 using two real bases $\beta_0, \beta_1$ alternatively:

$$\beta_0 = \lim_{n\to+\infty} \frac{U_{2n}}{U_{2n-1}} = \frac{5+\sqrt{13}}{6} \quad \text{and} \quad \beta_1 = \lim_{n\to+\infty} \frac{U_{2n-1}}{U_{2n-2}} = \frac{1+\sqrt{13}}{2}.$$

- In the previous example, the polynomial of the recurrence relation satisfied by $U$ was of the form $P(x^2)$ for some polynomial $P$. This does not have to be the case, contradicting Hollander's intuition.

  Consider $U = (U_n)_{n\geq 0}$ starting with $1, 3, 8$ and such that $U_n = 2U_{n-1} - 4U_{n-2} + 8U_{n-3}$ for $n \geq 3$. We get $U = (1, 3, 8, 12, 16, 48, 128, 192, 256, \ldots)$.

  Again, this system has no dominant root but the numeration language is regular:

| $U_n - 1$ | 0 | 2 | 7 | 11 | 15 | 47 | 127 | 191 | 255 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{rep}_U(U_n - 1)$ | $\varepsilon$ | 2 | 21 | 110 | 1010 | 21010 | 211010 | 1101010 | 10101010 | $\cdots$ |

  Here the maximal words follow 4 infinite words, which are $2(10)^\omega$, $21(10)^\omega$, $(10)^\omega$ and $1(10)^\omega$.

  This time, to understand these words, we have to represent 1 using four real bases $\beta_0, \beta_1, \beta_2, \beta_3$ alternatively:

$$\beta_0 = \lim_{n\to+\infty} \frac{U_{4n}}{U_{4n-1}} = \frac{4}{3}, \qquad \beta_1 = \lim_{n\to+\infty} \frac{U_{4n-1}}{U_{4n-2}} = \frac{3}{2},$$
$$\beta_2 = \lim_{n\to+\infty} \frac{U_{4n-2}}{U_{4n-3}} = \frac{8}{3}, \qquad \beta_3 = \lim_{n\to+\infty} \frac{U_{4n-3}}{U_{4n-4}} = 3.$$

# Getting rid of the dominant root condition

First step: Exploit the positiveness of the generating series.

- If $L_U$ is regular, then the series $\sum_{n\geq 0} U_n X^n$ is $\mathbb{N}$-rational (and not just $\mathbb{Z}$-rational).
- From a result of Berstel from 1971, we obtain that if a series $\sum_{i\geq 0} s_n X^n$ is $\mathbb{N}$-rational, then there exists some $p \geq 1$ such that for each $i \in \{0, \ldots, p-1\}$, the limit

$$\lim_{n\to+\infty} \frac{s_{pn-i}}{s_{pn-i-1}}$$

  exists.
- Consequently, if $L_U$ is regular, then we can associate with $U$ a $p$-tuple of real numbers $(\beta_0, \ldots, \beta_{p-1})$ where for each $i$,

$$\beta_i := \lim_{n\to+\infty} \frac{U_{pn-i}}{U_{pn-i-1}}.$$

# Alternate bases

Second step: Introduce alternate bases and link them with maximal words of each length in $L_U$.

- For a tuple $B = (\beta_0, \dots, \beta_{p-1})$, we consider representations of real numbers of the form

$$x = \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \frac{a_2}{\beta_0\beta_1\beta_2} + \cdots$$

  where $\beta_n := \beta_{n \bmod p}$ for all $n \geq 0$.

- We use a greedy algorithm to define greedy $B$-expansions of real numbers $d_B(x)$.

- We define the quasi-greedy $B$-expansion of 1 as $d_B^*(1) = \lim_{x \to 1^-} d_B(x)$.

- We get a Parry-kind characterization of allowable sequences: a sequence $a_0a_1a_2\cdots$ is the $B$-expansion of a real number in $[0, 1)$ if and only if for all $n \geq 0$, one has $a_na_{n+1}a_{n+2}\cdots <_{\text{lex}} d_{S^n(B)}^*(1)$ where $S^n(B)$ is the shifted base $(\beta_n, \beta_{n+1}, \dots)$.

[C-Cisternino 2021]

- If we take the alternate base $B = (\beta_0, \beta_1) = (\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2})$, which was obtained in the previous examples of positional numeration system, we can compute

$$d_B(1) = 110^\omega \quad \text{and} \quad d_{S(B)}(1) = 2010^\omega.$$

In particular,

$$1 = \frac{1}{\beta_0} + \frac{1}{\beta_0\beta_1} \quad \text{and} \quad 1 = \frac{2}{\beta_1} + \frac{1}{\beta_1^2\beta_0}.$$

The quasi-greedy expansions of 1 are then given by

$$d_B^*(1) = (10)^\omega \quad \text{and} \quad d_{S(B)}^*(1) = 200(10)^\omega = 20(01)^\omega.$$

These are the two words followed by the maximal words in the numeration language that we have seen before.

This example shows a behavior similar to Parry real bases, and also to canonical Bertrand systems.

- If we take the alternate base $B = (\beta_0, \beta_1, \beta_2, \beta_3) = (\frac{4}{3}, \frac{3}{2}, \frac{8}{3}, 3)$, which was obtained in the previous examples of positional numeration system, we can compute

$$d_B(1) = 10110^\omega, \qquad d_{S(B)}(1) = 1110^\omega, \qquad d_{S(B)}(1) = 220^\omega, \qquad d_{S(B)}(1) = 30^\omega.$$

The quasi-greedy expansions of 1 are then given by

$$d^*_B(1) = (1010)^\omega, \qquad d^*_{S(B)}(1) = 1(1010)^\omega, \qquad d^*_{S^2(B)}(1) = 21(1010)^\omega, \qquad d^*_{S^3(B)}(1) = 2(1010)^\omega.$$

These are the four words followed by the maximal words in the numeration language that we have seen before.

This example again shows a behavior similar to Parry real bases and to canonical Bertrand systems, but without a dominant root.

## Lemma

*Let* $U = (U_n)_{n \geq 0}$ *be a positional numeration system with an associated alternate base* $(\beta_0, \dots, \beta_{p-1})$*. For all* $i \in \{0, \dots, p-1\}$*, all lengths* $L$ *and all large enough indices* $n$*, there exists* $j$ *such that*

$$\mathrm{Pref}_L(\mathrm{rep}_U(U_{pn-i} - 1)) = \mathrm{Pref}_L(\mathbf{w}_{i,j})$$

*where the infinite words* $\mathbf{w}_{i,j}$ *are* $(\beta_i, \dots, \beta_{i+p-1})$*-representations of* $1$ *which are "intermediate" between the greedy and the quasi-greedy one.*

## Proposition

*Let* $U$ *be a positional numeration system with a regular numeration language* $L_U$*, and let* $(\beta_0, \dots, \beta_{p-1})$ *be an associated alternate base.*
*Then for each* $i \in \{0, \dots, p-1\}$*, the quasi-greedy expansion* $d^*_{S^i(B)}(1)$ *is ultimately periodic.*

Such alternate bases are called Parry.

[C-Kreczman 2025+]

# Intermediate representations of 1

Let $U = (U_n)_{n\geq 0}$ be defined by $U_{n+10} = 16U_{n+5} - 9U_n$ for $n \geq 0$ and the following initial conditions.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $U_n$ | 1 | 2 | 3 | 6 | 10 | 19 | 29 | 48 | 96 | 151 |

Then for $i \in \{0, \ldots, 4\}$, the limits

$$\beta_i := \lim_{n\to+\infty} \frac{U_{5n-i}}{U_{5n-i-1}}$$

exist, and can be effectively computed.

Set $B = (\beta_0, \ldots, \beta_4)$.

We get the following greedy and quasi-greedy $S^i(B)$-expansions of 1:

| $i$ | $d_{S^i(B)}(1)$ |
|---|---|
| 0 | $1110^\omega$ |
| 1 | $11000(10000)^\omega$ |
| 2 | $20^\omega$ |
| 3 | $110^\omega$ |
| 4 | $110^\omega$ |

| $i$ | $d^*_{S^i(B)}(1)$ |
|---|---|
| 0 | $(11010)^\omega$ |
| 1 | $11000(10000)^\omega$ |
| 2 | $1(10110)^\omega$ |
| 3 | $(10110)^\omega$ |
| 4 | $1011000(10000)^\omega$ |

For $i = 0$, the intermediate representations are given by

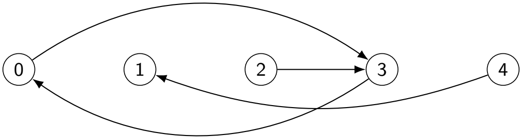
$$\mathbf{w}_{0,1} = 110 \cdot 110^\omega$$

$$\mathbf{w}_{0,2} = 110 \cdot 10 \cdot 1110^\omega$$

$$\mathbf{w}_{0,3} = 110 \cdot 10 \cdot 110 \cdot 110^\omega$$

$$\vdots$$

We encode the possible interactions of the remainders $i \in \{0, \ldots, p-1\}$ in a graph $G$:



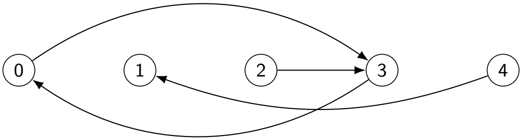

| $i$ | $d_{S^i(B)}(1)$ |
|---|---|
| 0 | $1110^\omega$ |
| 1 | $11000(10000)^\omega$ |
| 2 | $20^\omega$ |
| 3 | $110^\omega$ |
| 4 | $110^\omega$ |

| $i$ | $d^*_{S^i(B)}(1)$ |
|---|---|
| 0 | $(11010)^\omega$ |
| 1 | $11000(10000)^\omega$ |
| 2 | $1(10110)^\omega$ |
| 3 | $(10110)^\omega$ |
| 4 | $1011000(10000)^\omega$ |

Third step: Suppose that $U$ is a positional numeration system with an associated Parry alternate base $(\beta_0, \ldots, \beta_{p-1})$. Then study the regularity of the sub-languages

$$\begin{aligned} L_{U,i} &:= \{w \in \mathrm{Max}(L_U) : |w| \equiv -i \pmod{p}\} \\ &= \{\mathrm{rep}_U(U_{pn-i} - 1) : n \geq 1\} \end{aligned}$$

by analyzing all possible interactions between remainders as encoded in the graph $G$.




Note that, in the dominant root case, the graph $G$ reduces to two simple cases:

Thank you!