Alternate Bases: combinatorial, ergodic and algebraic properties

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Journées du GT SDA2, Toulouse, France
2023, March 29-31

Cantor real bases and alternate bases

A Cantor real base is a sequence $\boldsymbol{\mathcal{B}} = (\beta_n)_{n\geq 0}$ of real numbers such that

- $\beta_n > 1$ for all n
- $\prod_{n=0}^{\infty} \beta_n = \infty$.

A $\boldsymbol{\mathcal{B}}$-representation of a real number x is an infinite sequence $a = (a_n)_{n\geq 0}$ of integers such that

$$x = \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \frac{a_2}{\beta_0\beta_1\beta_2} + \cdots$$

In this case, we write $\mathrm{val}_{\boldsymbol{\mathcal{B}}}(a) = x$.

For $x \in [0,1]$, a distinguished $\boldsymbol{\mathcal{B}}$-representation

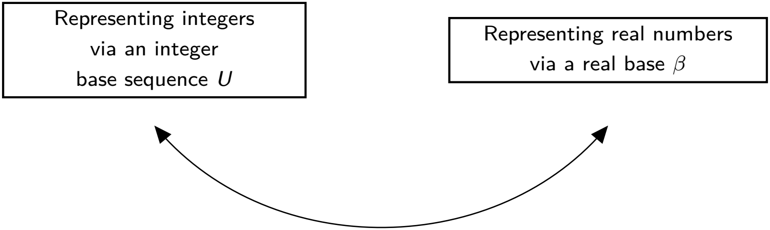
$$d_{\boldsymbol{\mathcal{B}}}(x) = (\varepsilon_n)_{n\geq 0},$$

called the $\boldsymbol{\mathcal{B}}$-expansion of x, is obtained from the greedy algorithm:

- We first set $r_0 = x$.
- Then set $\varepsilon_n = \lfloor \beta_n r_n \rfloor$ and $r_{n+1} = \beta_n r_n - \varepsilon_n$ for $n \geq 0$.

An alternate base is a periodic Cantor base. In this case, we simply write $\boldsymbol{\mathcal{B}} = (\beta_0, \ldots, \beta_{p-1})$ and we use the convention that $\beta_n = \beta_{n \bmod p}$ for all $n \geq 0$.

Motivation



Bertrand-Mathis's work

$$\frac{U_{n+1}}{U_n} \to \beta$$

When $\frac{U_{n+p}}{U_n} \to \beta$, there is a similar relationship with representations of real numbers via some alternate base $\boldsymbol{\mathcal{B}} = (\beta_0, \ldots, \beta_{p-1})$.

Let's look at a few examples

- The sequence $\boldsymbol{\mathcal{B}} = (1 + \frac{1}{2^{n+1}})_{n\geq 0}$ is not a Cantor real base since $\prod_{n=0}^{\infty} \alpha_n < \infty$. If we perform the greedy algorithm on $x = 1$, we obtain the sequence of digits 10^{ω}, which is clearly not a $\boldsymbol{\mathcal{B}}$-representation of 1.
- The sequence $\boldsymbol{\mathcal{B}} = (2 + \frac{1}{2^{n+1}})_{n\geq 0}$ is a Cantor real base since $\prod_{n=0}^{\infty} \alpha_n = \infty$.
- Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$. Consider the alternate base $\boldsymbol{\mathcal{B}} = (\alpha, \beta)$. Then $d_{\boldsymbol{\mathcal{B}}}(1) = 2010^{\omega}$.

$r_0 = 1$	$\varepsilon_0 = \lfloor \alpha r_0 \rfloor = \left\lfloor \frac{1+\sqrt{13}}{2} \right\rfloor = 2$
$r_1 = \alpha r_0 - \varepsilon_0 = \frac{-3+\sqrt{13}}{2}$	$\varepsilon_1 = \lfloor \beta r_1 \rfloor = \left\lfloor \frac{-1+\sqrt{13}}{6} \right\rfloor = 0$
$r_2 = \beta r_1 - \varepsilon_1 = \frac{-1+\sqrt{13}}{6}$	$\varepsilon_2 = \lfloor \alpha r_2 \rfloor = \lfloor 1 \rfloor = 1$
$r_3 = \alpha r_2 - \varepsilon_2 = 0$	$\varepsilon_3 = \lfloor \beta r_3 \rfloor = \lfloor 0 \rfloor = 0$

- Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$.
Let now $\mathcal{B} = (\beta_n)_{n\geq 0} = (\alpha, \beta, \beta, \alpha, \ldots)$ be the Thue-Morse sequence over $\{\alpha, \beta\}$:

$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise.} \end{cases}$$

We compute $d_{\mathcal{B}}(1) = 20010110^{\omega}$.

$r_0 = 1$	$\varepsilon_0 = \lfloor \alpha r_0 \rfloor = \lfloor \alpha \rfloor = 2$
$r_1 = \alpha r_0 - \varepsilon_0 = \frac{-3+\sqrt{13}}{2}$	$\varepsilon_1 = \lfloor \beta r_1 \rfloor = \left\lfloor \frac{-1+\sqrt{13}}{6} \right\rfloor = 0$
$r_2 = \beta r_1 - \varepsilon_1 = \frac{-1+\sqrt{13}}{6}$	$\varepsilon_2 = \lfloor \beta r_2 \rfloor = \left\lfloor \frac{2+\sqrt{13}}{9} \right\rfloor = 0$
$r_3 = \beta r_2 - \varepsilon_2 = \frac{2+\sqrt{13}}{9}$	$\varepsilon_3 = \lfloor \alpha r_3 \rfloor = \left\lfloor \frac{5+\sqrt{13}}{6} \right\rfloor = 1$
$r_4 = \alpha r_3 - \varepsilon_3 = \frac{-1+\sqrt{13}}{6}$	$\varepsilon_4 = \lfloor \beta r_4 \rfloor = \left\lfloor \frac{2+\sqrt{13}}{9} \right\rfloor = 0$
$r_5 = \beta r_4 - \varepsilon_4 = \frac{2+\sqrt{13}}{9}$	$\varepsilon_5 = \lfloor \alpha r_5 \rfloor = \left\lfloor \frac{5+\sqrt{13}}{6} \right\rfloor = 1$
$r_6 = \alpha r_5 - \varepsilon_5 = \frac{-1+\sqrt{13}}{6}$	$\varepsilon_6 = \lfloor \alpha r_6 \rfloor = \lfloor 1 \rfloor = 1$
$r_7 = \alpha r_6 - \varepsilon_6 = 0$	$\varepsilon_7 = \lfloor \beta r_7 \rfloor = \lfloor 0 \rfloor = 0$

- Consider the alternate base $\mathcal{B} = (\sqrt{6}, 3, \frac{2+\sqrt{6}}{3})$. Then $d_{\mathcal{B}}(1) = 2(10)^\omega$.

$r_0 = 1$	$\varepsilon_0 = \lfloor \sqrt{6} r_0 \rfloor = \lfloor \sqrt{6} \rfloor = 2$
$r_1 = \sqrt{6} r_0 - \varepsilon_0 = -2 + \sqrt{6}$	$\varepsilon_1 = \lfloor 3 r_1 \rfloor = \lfloor -6 - 3\sqrt{6} \rfloor = 1$
$r_2 = 3 r_1 - \varepsilon_1 = -7 + 3\sqrt{6}$	$\varepsilon_2 = \left\lfloor \frac{2+\sqrt{6}}{3} r_2 \right\rfloor = \left\lfloor \frac{4-\sqrt{6}}{3} \right\rfloor = 0$
$r_3 = \frac{2+\sqrt{6}}{3} r_2 - \varepsilon_2 = \frac{4-\sqrt{6}}{3}$	$\varepsilon_3 = \lfloor \sqrt{6} r_3 \rfloor = \left\lfloor \frac{-6+4\sqrt{6}}{3} \right\rfloor = 1$
$r_4 = \sqrt{6} r_3 - \varepsilon_3 = \frac{-9+4\sqrt{6}}{3}$	$\varepsilon_4 = \lfloor 3 r_4 \rfloor = \lfloor -9 + 4\sqrt{6} \rfloor = 0$
$r_5 = 3 r_4 - \varepsilon_4 = -9 + 4\sqrt{6}$	$\varepsilon_5 = \left\lfloor \frac{2+\sqrt{6}}{3} r_5 \right\rfloor = \left\lfloor \frac{6-\sqrt{6}}{3} \right\rfloor = 1$
$r_6 = \frac{2+\sqrt{6}}{3} r_5 - \varepsilon_5 = \frac{3-\sqrt{6}}{3}$	$\varepsilon_6 = \lfloor \sqrt{6} r_6 \rfloor = \lfloor -2 + \sqrt{6} \rfloor = 0$
$r_7 = \frac{2+\sqrt{6}}{3} r_6 - \varepsilon_6 = -2 + \sqrt{6}$	$\varepsilon_7 = \lfloor 3 r_7 \rfloor = \lfloor -6 - 3\sqrt{6} \rfloor = 1$

Parry's theorem for Cantor real bases

Recall Parry's theorem for real bases $\beta > 1$:

Theorem (Parry 1960)

A sequence $a_0 a_1 a_2 \cdots$ of non-negative integers is the β-expansion of some $x \in [0,1)$ if and only if $a_n a_{n+1} a_{n+2} \cdots <_{\text{lex}} d^*_\beta(1)$ for all n.

Here $d^*_\beta(1)$ is the quasi-greedy β-expansion of 1:

$$d^*_\beta(1) = \lim_{x \to 1^-} d_\beta(x).$$

Theorem (Charlier & Cisternino 2021)

A sequence $a_0 a_1 a_2 \cdots$ of non-negative integers is the $\boldsymbol{\mathcal{B}}$-expansion of some $x \in [0,1)$ if and only if $a_n a_{n+1} a_{n+2} \cdots <_{\text{lex}} d^*_{\boldsymbol{\mathcal{B}}^{(n)}}(1)$ for all n.

Here we use all shifted Cantor real bases

$$\boldsymbol{\mathcal{B}}^{(n)} = (\beta_n, \beta_{n+1}, \beta_{n+2}, \ldots)$$

and the quasi-greedy $\boldsymbol{\mathcal{B}}$-expansions of 1 are defined by

$$d^*_{\boldsymbol{\mathcal{B}}}(1) = \lim_{x \to 1^-} d_{\boldsymbol{\mathcal{B}}}(x).$$

Quasi-greedy expansions of 1

For real bases β, the quasi-greedy $\boldsymbol{\mathcal{B}}$-expansion of 1 is given by the formula

$$d^*_\beta(1) = \begin{cases} d_{\boldsymbol{\mathcal{B}}}(1) & \text{if } d_\beta(1) \text{ is infinite} \\ (\varepsilon_0 \cdots \varepsilon_{n-2}(\varepsilon_{n-1}-1))^\omega & \text{if } d_\beta(1) = \varepsilon_0 \cdots \varepsilon_{n-1} 0^\omega \text{ with } \varepsilon_{n-1} > 0. \end{cases}$$

For Cantor real base $\boldsymbol{\mathcal{B}}$, we have the following recursive way to compute the quasi-greedy $\boldsymbol{\mathcal{B}}$-expansion of 1:

$$d^*_{\boldsymbol{\mathcal{B}}}(1) = \begin{cases} d_{\boldsymbol{\mathcal{B}}}(1) & \text{if } d_{\boldsymbol{\mathcal{B}}}(1) \text{ is infinite} \\ \varepsilon_0 \cdots \varepsilon_{n-2}(\varepsilon_{n-1}-1) d^*_{\boldsymbol{\mathcal{B}}^{(n)}}(1) & \text{if } d_{\boldsymbol{\mathcal{B}}}(1) = \varepsilon_0 \cdots \varepsilon_{n-1} 0^\omega \text{ with } \varepsilon_{n-1} > 0. \end{cases}$$

On an example

Consider the alternate base $\boldsymbol{\mathcal{B}} = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$.

Then

$$d_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 2010^\omega \text{ and } d_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = 110^\omega.$$

We can compute

$$d^*_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 200(10)^\omega = 20(01)^\omega \text{ and } d^*_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = (10)^\omega.$$

By the previous theorem, the infinite sequence

$$20001001010020(001)^\omega$$

is the $\boldsymbol{\mathcal{B}}$-expansion of some $x \in [0, 1)$, whereas the infinite sequence

$$2000100110020(001)^\omega$$

isn't.

Combinatorial criteria for being the β-expansion of 1

As a consequence of his theorem, Parry obtained a combinatorial criteria for being the β-expansion of 1:

Theorem (Parry 1960)

A β-representation $a_0 a_1 a_2 \cdots$ of 1 is its β-expansion if and only if $a_n a_{n+1} a_{n+2} \cdots <_{\mathrm{lex}} a_0 a_1 a_2 \cdots$ for all $n \geq 1$.

What we can deduce for Cantor real bases is:

Theorem (Charlier & Cisternino 2021)

A $\boldsymbol{\mathcal{B}}$-representation $a_0 a_1 a_2 \cdots$ of 1 is its $\boldsymbol{\mathcal{B}}$-expansion if and only if $a_n a_{n+1} a_{n+2} \cdots <_{\mathrm{lex}} d^*_{\boldsymbol{\mathcal{B}}^{(n)}}(1)$ for all $n \geq 1$.

However, this result does not provide a purely combinatorial criteria for Cantor real bases, and this is true even for alternate bases.

A purely combinatorial condition for checking whether a $\mathcal{B}$-representation is greedy cannot exist

Given a sequence $a = a_0 a_1 a_2 \cdots$, there may exist more than one alternate base $\mathcal{B}$ such that $\mathrm{val}_{\mathcal{B}}(a) = 1$.

Among all of them, it may be that a is greedy for one and not greedy for another one:

- Consider $a = 2(10)^\omega$.
  Then $\mathrm{val}_{\mathcal{A}}(a) = \mathrm{val}_{\mathcal{B}}(a) = 1$ for both $\mathcal{A} = (1+\varphi, 2)$ and $\mathcal{B} = (\frac{31}{10}, \frac{420}{341})$.
  We can check that $d_{\mathcal{A}}(1) = a$, but $d_{\mathcal{B}}(1) \neq a$ since the first digit of $d_{\mathcal{A}}(1)$ is $\left\lfloor \frac{31}{10} \right\rfloor = 3$.

A sequence a can be greedy for more than one alternate base:

- The sequence 110^ω is the $\mathcal{B}$-expansion of 1 w.r.t φ, $(\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2})$ and $(\frac{17}{10}, \frac{10}{7})$.

At the opposite, it may happen that a sequence a is a representation of 1 for several alternate bases $\mathcal{B}$ but that none of these are such that a is greedy.

- The sequence $(10)^\omega$ is a $\mathcal{B}$-representation of 1 for the previous 3 alternate bases.
  Being periodic, it cannot be the $\mathcal{B}$-expansion of 1 for any alternate base.

Alternate $\mathcal{B}$-shift

For $\beta > 1$, the β-shift is defined as the topological closure of the set $\{d_\beta(x) \colon x \in [0,1)\}$.

Theorem (Bertrand-Mathis 1986)

The β-shift is sofic if and only if $d^*_\beta(1)$ is ultimately periodic.

For an alternate base $\mathcal{B}$, the set $\{d_{\mathcal{B}}(x) \colon x \in [0,1)\}$ is not shift-invariant in general.

The $\mathcal{B}$-shift is defined as the topological closure of the set

$$\bigcup_{i=0}^{p-1} \{d_{\mathcal{B}^{(i)}}(x) \colon x \in [0,1)\}.$$

Theorem (Charlier & Cisternino 2021)

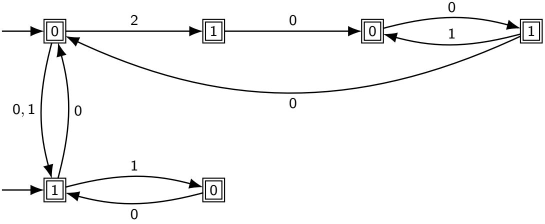
The $\mathcal{B}$-shift is sofic if and only if $d^*_{\mathcal{B}^{(i)}}(1)$ is ultimately periodic for all $i \in \{0,\ldots,p-1\}$.

In view of this result, we refer to such alternate bases as the Parry alternate bases.

Examples

For $\boldsymbol{\mathcal{B}} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$, we have $d^*_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 20(01)^\omega$ and $d^*_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = (10)^\omega$.

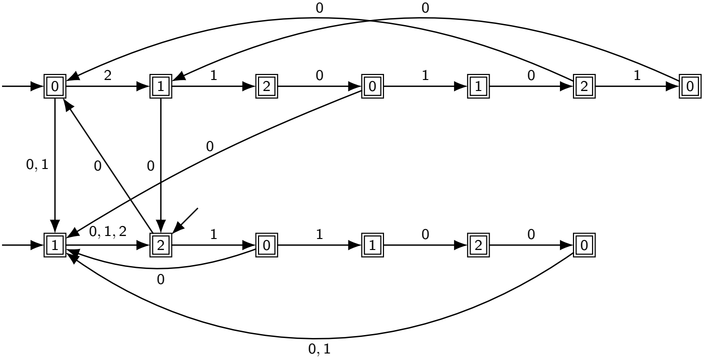
The following finite automaton accepts the set of factors of elements in the $\boldsymbol{\mathcal{B}}$-shift.



For $\mathcal{B} = (\sqrt{6}, 3, \frac{2+\sqrt{6}}{3})$, we have

$$d^*_{\mathcal{B}^{(0)}}(1) = 2(10)^\omega,\ d^*_{\mathcal{B}^{(1)}}(1) = (211001)^\omega,\ d^*_{\mathcal{B}^{(2)}}(1) = (110012)^\omega.$$

The following finite automaton accepts the set of factors of elements in the $\mathcal{B}$-shift.



Finite type?

A subshift S of $A^{\mathbb{N}}$ is said to be of finite type if its minimal set of forbidden factors is finite.

Theorem (Bertrand-Mathis 1986)

The β-shift is of finite type if and only if $d_\beta(1)$ is finite

However, this result does not generalize to alternate bases of length $p \geq 2$.

Indeed, for the alternate base $\boldsymbol{\mathcal{B}} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$, we have

$$d_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 2010^\omega \text{ and } d_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = 11^\omega.$$

Then

$$d^*_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 200(10)^\omega \text{ and } d^*_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = (10)^\omega$$

and we see that all words in $2(00)^*2$ are minimal forbidden factors, so the $\boldsymbol{\mathcal{B}}$-shift is not of finite type.

Necessary conditions on $\boldsymbol{\mathcal{B}}$ to be a Parry alternate base

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

If $\boldsymbol{\mathcal{B}} = (\beta_0, \ldots, \beta_{p-1})$ is a Parry alternate base and $\delta = \beta_0 \cdots \beta_{p-1}$, then

- *δ is an algebraic integer*
- *$\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \{0, \ldots, p-1\}$.*

Let me give some intuition on an example.

Let $\boldsymbol{\mathcal{B}} = (\beta_0, \beta_1, \beta_2)$ be a base such that the expansions of 1 are given by

$$d_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 30^\omega, \quad d_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = 110^\omega, \quad d_{\boldsymbol{\mathcal{B}}^{(2)}}(1) = 1(110)^\omega.$$

We derive that $\beta_0, \beta_1, \beta_2$ satisfy the following set of equations

$$\frac{3}{\beta_0} = 1, \quad \frac{1}{\beta_1} + \frac{1}{\beta_1\beta_2} = 1, \quad \frac{1}{\beta_2} + \left(\frac{1}{\beta_2\beta_0} + \frac{1}{\delta}\right)\frac{\delta}{\delta - 1} = 1,$$

where $\delta = \beta_0\beta_1\beta_2$.

Multiplying the first equation by δ, the second one by $\beta_1\beta_2$ and the third one by $(\delta - 1)\beta_2$, we obtain the identities

$$3\beta_1\beta_2 - \delta = 0, \quad -\beta_1\beta_2 + \beta_2 + 1 = 0, \quad \beta_1\beta_2 + (2-\delta)\beta_2 + \delta - 1 = 0.$$

In a matrix formalism, we have

$$\begin{pmatrix} 3 & 0 & -\delta \\ -1 & 1 & 1 \\ 1 & 2-\delta & \delta-1 \end{pmatrix} \begin{pmatrix} \beta_1\beta_2 \\ \beta_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The existence of a non-zero vector $(\beta_1\beta_2, \beta_2, 1)^T$ as a solution of this equation forces that the determinant of the coefficient matrix is zero:

$$\delta^2 - 9\delta + 9 = 0.$$

Hence we must have $\delta = \frac{9+3\sqrt{5}}{2} = 3\varphi^2$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

We then obtain

$$\beta_1\beta_2 = \frac{\delta}{3} = \varphi^2 \text{ and } \beta_2 = \beta_1\beta_2 - 1 = \varphi^2 - 1 = \varphi.$$

Consequently,

$$\beta_1 = \frac{\beta_1\beta_2}{\beta_2} = \frac{\varphi^2}{\varphi} = \varphi \text{ and } \beta_0 = \frac{\delta}{\beta_1\beta_2} = \frac{3\varphi^2}{\varphi^2} = 3.$$

Indeed, the triple $\boldsymbol{\mathcal{B}} = (3, \varphi, \varphi)$ is an alternate base giving precisely the given expansions of 1.

The same strategy can be applied to any Parry alternate base.

However, the product δ need not be a Parry number

One might think at first that the product $\delta = \beta_0 \cdots \beta_{p-1}$ should be a Parry number since by grouping terms p by p in the sum

$$\frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \frac{a_2}{\beta_0\beta_1\beta_2} + \cdots$$

we get an expansion of the kind

$$\frac{c_0}{\delta} + \frac{c_1}{\delta^2} + \frac{c_2}{\delta^3} + \cdots .$$

But here, the numerators are no longer integers.

Consider again the Parry alternate base $\boldsymbol{\mathcal{B}} = (3, \varphi, \varphi)$. Then the previous grouping for the expansions

$$d_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 30^\omega, \quad d_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = 110^\omega, \quad d_{\boldsymbol{\mathcal{B}}^{(2)}}(1) = 1(110)^\omega$$

gives us

$$1 = \frac{3\varphi^2}{\delta}, \quad 1 = \frac{3\varphi + 3}{\delta}, \quad 1 = \frac{3\varphi + \varphi + 1}{\delta} + \frac{\varphi + 1}{\delta^2} + \frac{\varphi + 1}{\delta^3} + \frac{\varphi + 1}{\delta^4} + \cdots$$

In fact, we can show that $\delta = 3\varphi^2$ is not a Parry number, and moreover, none of its powers $\delta^n = (3\varphi^2)^n$ is.

A sufficient condition on $\boldsymbol{\mathcal{B}}$ to be a Parry alternate base

Let

- $\delta = \beta_0 \cdots \beta_{p-1}$
- $\boldsymbol{D} = (D_0, \ldots, D_{p-1})$ be a p-tuple of alphabets of integers containing 0
- $\mathcal{D} = \left\{ \sum_{i=0}^{p-1} a_i \beta_{i+1} \cdots \beta_{p-1} : a_i \in D_i \right\}$ be the corresponding set of numerators when grouping terms p by p
- $X^{\mathcal{D}}(\delta) = \left\{ \sum_{i=0}^{\ell-1} c_i \delta^{\ell-1-i} : \ell \in \mathbb{N},\ c_i \in \mathcal{D} \right\}$ be the associated complex spectrum.

Proposition

If $D_i \supseteq \{-\lfloor \beta_i \rfloor, \ldots, \lfloor \beta_i \rfloor\}$ for all $i \in \{0, \ldots, p-1\}$ and if the spectrum $X^{\mathcal{D}}(\delta)$ has no accumulation point in $\mathbb{R}$, then $\boldsymbol{\mathcal{B}}$ is a Parry alternate base.

Proposition

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then the spectrum $X^{\mathcal{D}}(\delta)$ has no accumulation point in $\mathbb{R}$.

As a consequence, we get

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then $\boldsymbol{\mathcal{B}}$ is a Parry alternate base.

Some remarks

- The condition of δ being a Pisot number is neither sufficient nor necessary for $\boldsymbol{\mathcal{B}}$ to be a Parry alternate base.
  1. Even for $p = 1$, there exist Parry numbers which are not Pisot.
  2. To see that it is not sufficient for $p \geq 2$, consider the alternate base $\boldsymbol{\mathcal{B}} = (\sqrt{\beta}, \sqrt{\beta})$ where β is the smallest Pisot number. The product δ is the Pisot number β. However, the $\boldsymbol{\mathcal{B}}$-expansion of 1 is equal to $d_{\sqrt{\beta}}(1)$, which is aperiodic.
- The bases $\beta_0, \ldots, \beta_{p-1}$ need not be algebraic integers in order to have a Parry alternate base.

  To see this, consider $\boldsymbol{\mathcal{B}} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. For this base, we have $d_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 2010^\omega$ and $d_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = 110^\omega$. However, the minimal polynomial of $\frac{5+\sqrt{13}}{6}$ is $3x^2 - 5x + 1$, hence it is not an algebraic integer.
- For the same non Pisot algebraic integer δ, there may exist a Parry alternate base $\boldsymbol{\alpha} = (\alpha_0, \cdots, \alpha_{p-1})$ and a non-Parry alternate base $\boldsymbol{\mathcal{B}} = (\beta_0 \cdots \beta_{p-1})$ such that $\prod_{i=0}^{p-1} \alpha_i = \prod_{i=0}^{p-1} \beta_i = \delta$.

Generalization of Schmidt's results

For $\beta > 1$, define $\mathrm{Per}(\beta) = \{x \in [0,1) : d_\beta(x) \text{ is ultimately periodic}\}$.

Theorem (Schmidt 1980)

1. *If $\mathbb{Q} \cap [0,1) \subseteq \mathrm{Per}(\beta)$ then β is either a Pisot number or a Salem number.*
2. *If β is a Pisot number then $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1)$.*

Define $\mathrm{Per}(\boldsymbol{\mathcal{B}}) = \{x \in [0,1) : d_{\boldsymbol{\mathcal{B}}}(x) \text{ is ultimately periodic}\}$.

Theorem (Charlier, Cisternino & Kreczman)

1. *If $\mathbb{Q} \cap [0,1) \subseteq \bigcap_{i=0}^{p-1} \mathrm{Per}(\boldsymbol{\mathcal{B}}^{(i)})$ then $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ and δ is either a Pisot number or a Salem number.*
2. *If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then $\mathrm{Per}(\boldsymbol{\mathcal{B}}) = \mathbb{Q}(\delta) \cap [0,1)$.*

From this, we recover the previously mentioned result (not using properties of the spectrum):

Corollary

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then $\boldsymbol{\mathcal{B}}$ is a Parry alternate base.

Theorem (Schmidt 1980)

If β *is an algebraic integer that is neither a Pisot number nor a Salem number then* $\mathrm{Per}(\beta) \cap \mathbb{Q}$ *is nowhere dense in* $[0, 1)$*.*

Theorem (Charlier, Cisternino & Kreczman)

If δ *is an algebraic integer that is neither a Pisot number nor a Salem number then* $\mathrm{Per}(\boldsymbol{\mathcal{B}}) \cap \mathbb{Q}$ *is nowhere dense in* $[0, 1)$*.*

Alternate zero automaton

For an alternate base $\boldsymbol{\mathcal{B}} = (\beta_0, \ldots, \beta_{p-1})$ and a p-tuple of alphabets $\boldsymbol{D} = (D_0, \ldots, D_{p-1})$, we can define a Büchi automaton $\mathcal{Z}(\boldsymbol{\mathcal{B}}, \boldsymbol{D})$ accepting the set

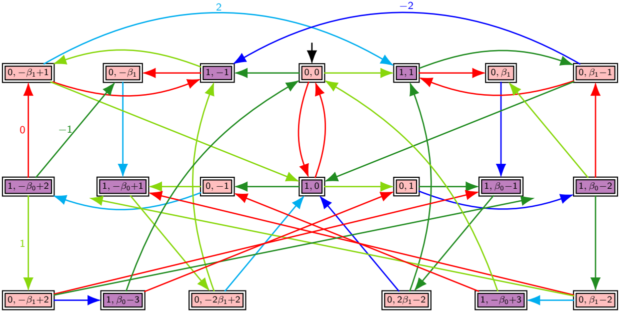
$$Z(\boldsymbol{\mathcal{B}}, \boldsymbol{D}) = \left\{ a_0 a_1 a_2 \cdots \in \prod_{n=0}^{+\infty} D_n \ : \ \sum_{n=0}^{+\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k} = 0 \right\}.$$

Here, we have set $D_n = D_{n \bmod p}$ and $\beta_n = \beta_{n \bmod p}$.

An example

Consider the alternate base $\boldsymbol{\mathcal{B}} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$ and $\boldsymbol{D} = (\{-2,-1,0,1,2\},\{-1,0,1\})$.

Then the zero automaton $\mathcal{Z}(\boldsymbol{\mathcal{B}}, \boldsymbol{D})$ is:



For instance, the infinite words $1(\overline{1}0)^\omega$ and $(0\overline{1}21\overline{21})^\omega$ have value 0 in base $\boldsymbol{\mathcal{B}}$ (where $\overline{1}$ and $\overline{2}$ designate the digits -1 and -2 respectively).

Alternate bases whose set of zero representations is accepted by a finite Büchi automaton

Theorem (Charlier, Cisternino, Masáková & Pelantová 2023)

The following assertions are equivalent.

1. *The zero automaton* $\mathcal{Z}(\boldsymbol{\mathcal{B}}, \boldsymbol{D})$ *is finite for all* $\boldsymbol{D} = (D_0, \ldots, D_{p-1})$.
2. *The zero automaton* $\mathcal{Z}(\boldsymbol{\mathcal{B}}, \boldsymbol{D})$ *is finite for one* $\boldsymbol{D} = (D_0, \ldots, D_{p-1})$ *such that*
   - $D_i \supseteq \{-\lfloor \beta_i \rfloor, \ldots, \lfloor \beta_i \rfloor\}$ *for all* i
   - $\lfloor \beta_i \rfloor \geq \lceil \delta \rceil - 1$ *for at least one* i.
3. *The product* δ *is a Pisot number and* $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$.

Normalization in alternate base

The normalization function is the partial function $\nu_{\mathcal{B},\boldsymbol{D}}$ mapping any $\mathcal{B}$-representation $a \in \prod_{n\in\mathbb{N}} D_n$ of a real number $x \in [0,1)$ to the $\mathcal{B}$-expansion of x.

We say that $\nu_{\mathcal{B},\boldsymbol{D}}$ is computable by a finite automaton if there exists a finite Büchi automaton accepting the set

$$\Big\{(u,v) \in \prod_{n\in\mathbb{N}} (D_n \times \{0,\ldots,\lceil\beta_n\rceil - 1\}) : \mathrm{val}_{\mathcal{B}}(u) = \mathrm{val}_{\mathcal{B}}(v) \text{ and } \exists x \in [0,1),\ v = d_{\mathcal{B}}(x)\Big\}.$$

First ingredient.

Consider two p-tuples of alphabets $\boldsymbol{D} = (D_0,\ldots,D_{p-1})$ and $\boldsymbol{D'} = (D'_0,\ldots,D'_{p-1})$.
We set $\boldsymbol{D} - \boldsymbol{D'} = (D_0 - D'_0,\ldots,D_{p-1} - D'_{p-1})$.

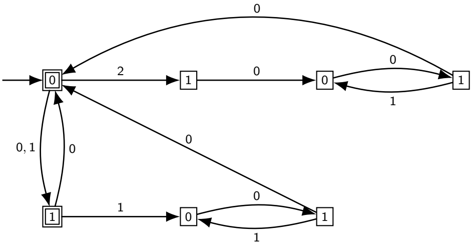
From the zero automaton $\mathcal{Z}(\mathcal{B},\boldsymbol{D}-\boldsymbol{D'})$, we define a converter $\mathcal{C}_{\mathcal{B},\boldsymbol{D},\boldsymbol{D'}}$ from $\boldsymbol{D}$ to $\boldsymbol{D}'$, that is, a Büchi automaton accepting the set

$$\{(u,v) \in \prod_{n\in\mathbb{N}} (D_n \times D'_n) : \mathrm{val}_{\mathcal{B}}(u) = \mathrm{val}_{\mathcal{B}}(v)\}.$$

Second ingredient.

In the case where $\boldsymbol{\mathcal{B}}$ is a Parry alternate base, we can define a Büchi automaton accepting the set $\{d_{\boldsymbol{\mathcal{B}}}(x) : x \in [0,1)\}$.

For $\boldsymbol{\mathcal{B}} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$, we have seen that $d^*_{\boldsymbol{\mathcal{B}}^{(0)}}(1) = 20(01)^\omega$ and $d^*_{\boldsymbol{\mathcal{B}}^{(1)}}(1) = (10)^\omega$.



Combining these two automata, we obtain the following result.

Theorem (Charlier, Cisternino, Masáková & Pelantová 2023)

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$, then the normalization function $\nu_{\boldsymbol{\mathcal{B}},\boldsymbol{D}}$ is computable by a finite Büchi automaton.

Ergodic properties of alternate base expansions

We can express the greedy digits a_n thanks to the β_n-transformations.

If $x \in [0,1)$ and $d_{\boldsymbol{\mathcal{B}}}(x) = a_0 a_1 a_2 \cdots$ then

$$a_n = \left\lfloor \beta_n \left(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x) \right) \right\rfloor$$

where for $\beta > 1$, the map

$$T_\beta \colon [0,1) \to [0,1), \ x \mapsto \beta x - \lfloor \beta x \rfloor .$$

is the so-called β-transformation.

A fundamental dynamical result of real base expansions is the following.

Theorem (Renyi 1957, Parry 1960, Rohlin 1961)

There exists a unique T_β-invariant absolutely continuous probability measure μ_β on $\mathcal{B}([0,1))$. Furthermore, the measure μ_β is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$ and the dynamical system $([0,1), \mathcal{B}([0,1)), \mu_\beta, T_\beta)$ is ergodic and has entropy $\log(\beta)$.

The alternate $\mathcal{B}$-transformation

Let $\mathcal{B} = (\beta_0, \ldots, \beta_{p-1})$ be an alternate base.

Then the $\mathcal{B}$-transformation is the map

$$T_{\mathcal{B}} \colon \{0, \ldots, p-1\} \times [0,1) \to \{0, \ldots, p-1\} \times [0,1), \ (i, x) \mapsto \big((i+1) \bmod p, T_{\beta_i}(x)\big) .$$

If $x \in [0,1)$ and $d_{\mathcal{B}}(x) = a_0 a_1 a_2 \cdots$ then

$$a_n = \lfloor \beta_n \pi_2 \left(T_{\mathcal{B}}^n(0, x)\right) \rfloor$$

for all $n \geq 0$, where π_2 is the projection on the second component.

The following proposition provides us with the main tool for the construction of a $T_{\boldsymbol{\mathcal{B}}}$-invariant measure.

Proposition (Charlier, Cisternino & Dajani 2023)

For all $n \geq 1$ and all $\beta_0, \ldots, \beta_{n-1} > 1$, there exists a unique $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$-invariant absolutely continuous probability measure μ on $\mathcal{B}([0,1))$. Furthermore, the measure μ is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$, and the associated dynamical system is exact and has entropy $\log(\beta_{n-1} \cdots \beta_0)$.

The probability measure $\mu_{\mathcal{B}}$

For each $i \in \{0, \ldots, p-1\}$, we let $\mu_{\mathcal{B},i}$ denote the unique $(T_{\beta_{i+p-1}} \circ \cdots \circ T_{\beta_i})$-invariant absolutely continuous probability measure.

We define a probability measure $\mu_{\mathcal{B}}$ on the σ-algebra

$$\mathcal{T}_p = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) : \forall i \in \{0, \ldots, p-1\},\ B_i \in \mathcal{B}([0,1)) \right\}$$

over $\{0, \ldots, p-1\} \times [0,1)$ as follows.

For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$, we set

$$\mu_{\mathcal{B}}\left(\bigcup_{i=0}^{p-1} (\{i\} \times B_i) \right) = \frac{1}{p} \sum_{i=0}^{p-1} \mu_{\mathcal{B},i}(B_i).$$

We define a new measure λ_p over the σ-algebra $\mathcal{T}_p$.

For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$, we set

$$\lambda_p\left(\bigcup_{i=0}^{p-1}(\{i\} \times B_i)\right) = \frac{1}{p}\sum_{i=0}^{p-1}\lambda(B_i).$$

We call this measure the p-Lebesgue measure on $\mathcal{T}_p$.

Theorem (Charlier, Cisternino & Dajani 2023)

The measure $\mu_{\boldsymbol{\mathcal{B}}}$ is the unique $T_{\boldsymbol{\mathcal{B}}}$-invariant probability measure on $\mathcal{T}_p$ that is absolutely continuous with respect to λ_p. Furthermore, $\mu_{\boldsymbol{\mathcal{B}}}$ is equivalent to λ_p on $\mathcal{T}_p$ and the dynamical system $(\{0,\ldots,p-1\} \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\mathcal{B}}}, T_{\boldsymbol{\mathcal{B}}})$ is ergodic and has entropy $\frac{1}{p}\log(\beta_0 \cdots \beta_{p-1})$.

Note that, however, the dynamical system $(\{0,\ldots,p-1\} \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\mathcal{B}}}, T_{\boldsymbol{\mathcal{B}}}^p)$ is not ergodic for $p > 1$.

Indeed, we have $T_{\boldsymbol{\mathcal{B}}}^{-p}(\{0\} \times [0,1)) = \{0\} \times [0,1)$ whereas $\mu_{\boldsymbol{\mathcal{B}}}(\{0\} \times [0,1)) = \frac{1}{p}$.

Frequencies of the digits

The frequency of a digit d occurring in the $\mathcal{B}$-expansion $a_0 a_1 a_2 \cdots$ of a real number x in $[0,1)$ is equal to

$$\lim_{n\to\infty} \frac{1}{n} \#\{0 \leq k < n : a_k = d\},$$

provided that this limit exists.

Proposition (Charlier, Cisternino & Dajani 2023)

For λ-almost all $x \in [0,1)$, the frequency of any digit d occurring in the greedy $\mathcal{B}$-expansion of x exists and is equal to

$$\frac{1}{p} \sum_{i=0}^{p-1} \mu_{\mathcal{B},i} \left(\left[\tfrac{d}{\beta_i}, \tfrac{d+1}{\beta_i} \right) \cap [0,1) \right).$$

Open problems

- Understand the $\mathcal{B}$-shifts of finite type for alternate base.
- Study of the $\mathcal{B}$-shift of well-chosen Cantor bases $\mathcal{B} = (\beta_n)_{n\geq 0}$.
- Could the $\mathcal{B}$-shift be sofic for "automatic" Cantor bases?
- Refinement of our result concerning the alternate spectrum.
- Compute the topological entropy.
- ...

Thank you!