Spectrum, Algebraicity and Normalization in Alternate Bases

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Cantor real bases and alternate bases

Let $\beta = (\beta_n)_{n\geq 0}$ be a sequence of real numbers greater than 1 and such that $\prod_{n=0}^{\infty} \beta_n$ is infinite.

A β-representation of a real number x is an infinite sequence $a = (a_n)_{n\geq 0}$ of integers such that

$$x = \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \frac{a_2}{\beta_0\beta_1\beta_2} + \cdots$$

An alternate base is a periodic Cantor base. In this case, we simply write $\beta = (\beta_0, \ldots, \beta_{p-1})$ and we use the convention that

- $\beta_n = \beta_{n \bmod p}$
- $\beta^{(n)} = (\beta_n, \ldots, \beta_{n+p-1})$

for all $n \geq 0$. We call the number p the length of the alternate base β.

Greedy algorithm

For $x \in [0,1]$, a distinguished β-representation

$$d_\beta(x) = (\varepsilon_n)_{n \geq 0},$$

called the β-expansion of x, is obtained from the greedy algorithm:

- $r_0 = x$
- $\varepsilon_n = \lfloor \beta_n r_n \rfloor$ and $r_{n+1} = \beta_n r_n - \varepsilon_n$ for $n \in \mathbb{N}$.

For each n, we have $\varepsilon_n \in \{0, 1, \ldots, \lfloor \beta_n \rfloor\}$.

Thus, the β-expansions are written over the alphabet $\{0, 1, \ldots, \max_{0 \leq i < p} \lfloor \beta_i \rfloor\}$.

Parry's theorem for alternate bases and alternate β-shift

The *quasi-greedy β-expansion of* 1 is $d^*_{\boldsymbol{\beta}}(1) = \lim_{x\to 1^-} d_{\boldsymbol{\beta}}(x)$.

Theorem (Charlier & Cisternino 2021)

An infinite sequence $a_0a_1a_2\cdots$ of non-negative integers belongs to the set $\{d_{\boldsymbol{\beta}}(x) \colon x \in [0,1)\}$ if and only if $a_na_{n+1}a_{n+2}\cdots <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$ for all $n \in \mathbb{N}$.

For an alternate base $\boldsymbol{\beta}$, the set $\{d_{\boldsymbol{\beta}}(x) \colon x \in [0,1)\}$ is not shift-invariant in general.

The $\boldsymbol{\beta}$-shift is defined as the topological closure of the set

$$\bigcup_{i=0}^{p-1} \{d_{\boldsymbol{\beta}^{(i)}}(x) \colon x \in [0,1)\}.$$

Theorem (Charlier & Cisternino 2021)

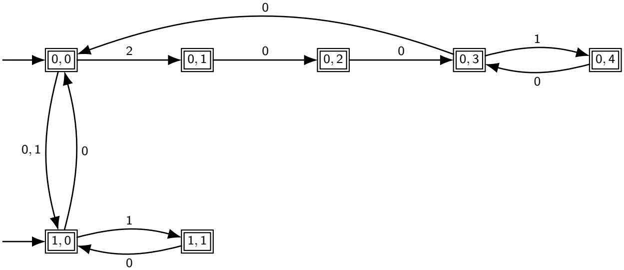
The $\boldsymbol{\beta}$-shift is sofic if and only if $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is eventually periodic for all $i \in \{0,\ldots,p-1\}$.

In view of this result, we refer to such alternate bases as the Parry alternate bases.

Example

Let $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We can compute $d^*_{\beta^{(0)}}(1) = 200(10)^\omega$ and $d^*_{\beta^{(1)}}(1) = (10)^\omega$.

The following finite automaton accepts the set of factors of elements in the β-shift.



Aims of this work

- Algebraic properties of Parry alternate bases.
  1. A necessary condition for being a Parry alternate base is that the product $\delta = \prod_{i=0}^{p-1} \beta_i$ is an algebraic integer and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$.
  2. A sufficient condition for being a Parry alternate base if that δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$.
- Normalization of alternate base representations.

  If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$, then the normalization function is computable by a finite Büchi automaton. Such an automaton is effectively given.

Spectrum as a tool

The notion of spectrum associated with a real base $\beta > 1$ and an alphabet of the form $A_d = \{0, 1, \ldots, d\}$ with $d \in \mathbb{N}$ was introduced by Erdős, Joó and Komornik in 1990.

For our purposes, we use a generalized concept of complex spectrum, and study its topological properties.

Let $\delta \in \mathbb{C}$ such that $|\delta| > 1$ with an alphabet $A \subset \mathbb{C}$.

The spectrum associated with δ and A is the set

$$X^A(\delta) = \left\{ \sum_{i=0}^{\ell-1} a_i \delta^{\ell-1-i} : n \in \mathbb{N},\ a_i \in A \right\}.$$

We say that a word $a_0 \cdots a_{\ell-1}$ over A corresponds to the element $\sum_{i=0}^{\ell-1} a_i \delta^{\ell-1-i}$ in the spectrum $X^A(\delta)$.

The following result shows that topological properties of the spectrum are linked with arithmetical aspects of the numeration system.

Theorem (Frougny & Pelantová 2018)

Let $\beta > 1$ *and* $d \in \mathbb{N}$*. Then* $Z(\beta, d)$ *is accepted by a finite Büchi automaton if and only if the spectrum* $X^d(\beta)$ *has no accumulation point in* $\mathbb{R}$*.*

For the case of real bases and symmetric integer alphabets, there is a complete characterization of the bases which give spectra without accumulation points in dependence on the alphabet.

Theorem (Akiyama & Komornik 2013, Feng 2016)

Let $\beta > 1$ *and* $d \in \mathbb{N}$*. The spectrum* $X^d(\beta)$ *has no accumulation point in* $\mathbb{R}$ *if and only if either* $\beta - 1 \geq d$ *or* β *is a Pisot number.*

Set of δ-representations of zero and complex zero automaton

For a complex base δ and an alphabet A of complex numbers, we define

$$Z(\delta, A) = \{a \in A^{\mathbb{N}} : \sum_{n=0}^{+\infty} \frac{a_n}{\delta^{n+1}} = 0\}.$$

Generalizing ideas from Frougny, we define a Büchi automaton

$$\mathcal{Z}(\delta, A) = (Q, 0, Q, A, E).$$

- States: $Q = X^A(\delta) \cap \{z \in \mathbb{C} : |z| \leq \frac{M}{|\delta|-1}\}$ where $M = \max\{|a| : a \in A\}$.
- Transitions: $E = \{(z, a, z\delta + a) : z \in Q,\ a \in A\}$.

Proposition

The Büchi automaton $\mathcal{Z}(\delta, A)$ accepts the set $Z(\delta, A)$.

Linking the complex spectrum and the complex zero automaton

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

Let δ be a complex number such that $|\delta| > 1$ and let A be an alphabet of complex numbers. Then the following assertions are equivalent.

1. *The set $Z(\delta, A)$ is accepted by a finite Büchi automaton.*
2. *The zero automaton $\mathcal{Z}(\delta, A)$ is finite.*
3. *The spectrum $X^A(\delta)$ has no accumulation point in $\mathbb{C}$.*

Towards an analogous result for alternate bases

- We consider a fixed alternate base $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$.
- We set $\delta = \prod_{i=0}^{p-1} \beta_i$.
- We consider a p-tuple $\boldsymbol{D} = (D_0, \ldots, D_{p-1})$ where, for all $i \in \{0, \ldots, p-1\}$, D_i is an alphabet of integers containing 0.
- We use the convention that for all $n \in \mathbb{Z}$, $D_n = D_{n \bmod p}$ and $\boldsymbol{D}^{(n)} = (D_n, \ldots, D_{n+p-1})$.

Grouping terms p by p, the equality

$$x = \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \cdots + \frac{a_{p-1}}{\beta_0\beta_1\cdots\beta_{p-1}} + \cdots$$

can be written as

$$x = \frac{\sum_{i=0}^{p-1} a_i\beta_{i+1}\cdots\beta_{p-1}}{\delta} + \frac{\sum_{i=0}^{p-1} a_{p+i}\beta_{i+1}\cdots\beta_{p-1}}{\delta^2} + \cdots$$

If we add the constraint that each letter a_n belongs to D_n, then we obtain a δ-representation of x over the alphabet

$$\mathcal{D} = \left\{ \sum_{i=0}^{p-1} a_i\beta_{i+1}\cdots\beta_{p-1} : \forall i \in \{0, \ldots, p-1\},\ a_i \in D_i \right\}.$$

Alternate spectrum

For $\delta = \prod_{i=0}^{p-1} \beta_i$ and the alphabet $\mathcal{D}$, we consider the spectrum $X^{\mathcal{D}}(\delta)$.

For each $i \in \{0, \ldots, p-1\}$, we let $X(i)$ denote the spectrum built from the shifted base $\beta^{(i)}$ and the shifted p-tuple of alphabets $\boldsymbol{D}^{(i)}$.

In particular, we have $X(0) = X^{\mathcal{D}}(\delta)$.

Lemma

For each $i \in \{0, \ldots, p-1\}$, we have $X(i) \cdot \beta_i + D_i = X(i+1)$ where $X(p) = X(0)$.

Alternate zero automaton

For each $i \in \{0, \ldots, p-1\}$, we define

$$M^{(i)} = \sum_{n=i}^{+\infty} \frac{\max(D_n)}{\prod_{k=i}^{n} \beta_k} \qquad \text{and} \qquad m^{(i)} = \sum_{n=i}^{+\infty} \frac{\min(D_n)}{\prod_{k=i}^{n} \beta_k}.$$

We define a Büchi automaton associated with an alternate base β and a p-tuple of alphabets $\boldsymbol{D}$ as

$$\mathcal{Z}(\beta, \boldsymbol{D}) = (Q_{\beta,\boldsymbol{D}}, (0,0), Q_{\beta,\boldsymbol{D}}, \cup_{i=0}^{p-1} D_i, E)$$

where

- $Q_{\beta,\boldsymbol{D}} = \bigcup_{i=0}^{p-1}(\{i\} \times (X(i) \cap [-M^{(i)}, -m^{(i)}]))$
- E is the set of transitions defined as follows: for $(i,s), (j,t) \in Q_{\beta,\boldsymbol{D}}$ and $a \in \cup_{i=0}^{p-1} D_i$, there is a transition $(i,s) \xrightarrow{a} (j,t)$ if and only if $j \equiv i+1 \pmod{p}$, $a \in D_i$ and $t = \beta_i s + a$.

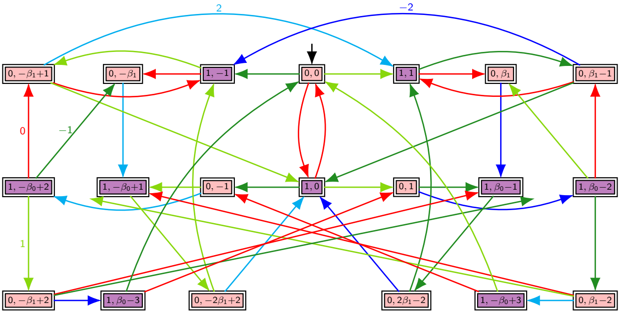
Proposition

The Büchi automaton $\mathcal{Z}(\beta, \boldsymbol{D})$ accepts the set

$$Z(\beta, \boldsymbol{D}) = \{a \in \prod_{n=0}^{+\infty} D_n \ : \ \sum_{n=0}^{+\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k} = 0\}.$$

An example

Consider the alternate base $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$ and $\boldsymbol{D} = (\{-2,-1,0,1,2\},\{-1,0,1\})$. Then $M^{(0)} = \mathrm{val}_{\boldsymbol{\beta}}((21)^\omega) \simeq 1.67994$ and $M^{(1)} = \mathrm{val}_{\boldsymbol{\beta}^{(1)}}((12)^\omega) \simeq 1.86852$.



Zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$

For instance, the infinite words $1(\overline{1}0)^\omega$ and $(0\overline{1}21\overline{2}\overline{1})^\omega$ have value 0 in base $\boldsymbol{\beta}$ (where $\overline{1}$ and $\overline{2}$ designate the digits -1 and -2 respectively).

Linking the alternate spectrum and the alternate zero automaton

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

Let β be an alternate base of length p and let $\boldsymbol{D}$ be a p-tuple of alphabets of integers containing 0. Then the following assertions are equivalent.

1. *The set $Z(\beta, \boldsymbol{D})$ is accepted by a finite Büchi automaton.*
2. *The zero automaton $\mathcal{Z}(\beta, \boldsymbol{D})$ is finite.*
3. *The spectrum $X^{\mathcal{D}}(\delta)$ has no accumulation point in $\mathbb{R}$.*

Necessary conditions on β to be a Parry alternate base

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

If $\boldsymbol{\beta}$ is a Parry alternate base, then

- δ *is an algebraic integer*
- $\beta_i \in \mathbb{Q}(\delta)$ *for all* $i \in \{0, \dots, p-1\}$.

Let me give some intuition on an example.
Let $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$ be a base such that the expansions of 1 are given by

$$d_{\boldsymbol{\beta}}(1) = 30^\omega, \quad d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega, \quad d_{\boldsymbol{\beta}^{(2)}}(1) = 1(110)^\omega.$$

We derive that $\beta_0, \beta_1, \beta_2$ satisfy the following set of equations

$$\frac{3}{\beta_0} = 1, \quad \frac{1}{\beta_1} + \frac{1}{\beta_1\beta_2} = 1, \quad \frac{1}{\beta_2} + \left(\frac{1}{\beta_2\beta_0} + \frac{1}{\delta}\right)\frac{\delta}{\delta-1} = 1,$$

where $\delta = \beta_0\beta_1\beta_2$.

Multiplying the first equation by δ, the second one by $\beta_1\beta_2$ and the third one by $(\delta-1)\beta_2$, we obtain the identities

$$3\beta_1\beta_2 - \delta = 0, \quad -\beta_1\beta_2 + \beta_2 + 1 = 0, \quad \beta_1\beta_2 + (2-\delta)\beta_2 + \delta - 1 = 0.$$

In a matrix formalism, we have

$$\begin{pmatrix} 3 & 0 & -\delta \\ -1 & 1 & 1 \\ 1 & 2-\delta & \delta-1 \end{pmatrix} \begin{pmatrix} \beta_1\beta_2 \\ \beta_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The existence of a non-zero vector $(\beta_1\beta_2, \beta_2, 1)^T$ as a solution of this equation forces that the determinant of the coefficient matrix is zero:

$$\delta^2 - 9\delta + 9 = 0.$$

Hence we must have $\delta = \frac{9+3\sqrt{5}}{2} = 3\varphi^2$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

We then obtain

$$\beta_1\beta_2 = \frac{\delta}{3} = \varphi^2 \text{ and } \beta_2 = \beta_1\beta_2 - 1 = \varphi^2 - 1 = \varphi.$$

Consequently,

$$\beta_1 = \frac{\beta_1\beta_2}{\beta_2} = \frac{\varphi^2}{\varphi} = \varphi \text{ and } \beta_0 = \frac{\delta}{\beta_1\beta_2} = \frac{3\varphi^2}{\varphi^2} = 3.$$

Indeed, the triple $\boldsymbol{\beta} = (3, \varphi, \varphi)$ is an alternate base giving precisely the given expansions of 1.

For obtaining the values $\beta_0, \beta_1, \beta_2$ from the known eventually periodic expansions we have used the fact that $\beta_0, \beta_1, \beta_2$ and $\delta = \beta_0\beta_1\beta_2$ are solutions of a system of polynomial equations in four unknowns x_0, x_1, x_2, y, in our case

$$\begin{cases} 3x_1x_2 - y &= 0 \\ -x_1x_2 + x_2 + 1 &= 0 \\ x_1x_2 + (2-y)x_2 + y - 1 &= 0 \\ x_1x_2x_3 &= y. \end{cases}$$

The solution of the system yielded that δ is a root of a monic polynomial with integer coefficients, i.e., is an algebraic integer.

The same strategy can be applied to any Parry alternate base.

A sufficient condition on $\boldsymbol{\beta}$ to be a Parry alternate base

As previously:

- $\delta = \prod_{i=0}^{p-1} \beta_i$
- $\boldsymbol{D} = (D_0, \ldots, D_{p-1})$ is a p-tuple of alphabets of integers containing 0
- $\mathcal{D}$ is the corresponding alphabet of real numbers.

Proposition

If $D_i \supseteq \{-\lfloor \beta_i \rfloor, \ldots, \lfloor \beta_i \rfloor\}$ for all $i \in \{0, \ldots, p-1\}$ and if the spectrum $X^{\mathcal{D}}(\delta)$ has no accumulation point in $\mathbb{R}$, then $\boldsymbol{\beta}$ is a Parry alternate base.

Proposition

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then the spectrum $X^{\mathcal{D}}(\delta)$ has no accumulation point in $\mathbb{R}$.

As a consequence, we get

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then $\boldsymbol{\beta}$ is a Parry alternate base.

Some remarks

- The condition of δ being a Pisot number is neither sufficient nor necessary for β to be a Parry alternate base.
  1. Even for $p = 1$, there exist Parry numbers which are not Pisot.
  2. To see that it is not sufficient for $p \geq 2$, consider the alternate base $\boldsymbol{\beta} = (\sqrt{\beta}, \sqrt{\beta})$ where β is the smallest Pisot number. The product δ is the Pisot number β. However, the $\boldsymbol{\beta}$-expansion of 1 is equal to $d_{\sqrt{\beta}}(1)$, which is aperiodic.
- The bases $\beta_0, \dots, \beta_{p-1}$ need not be algebraic integers in order to have a Parry alternate base.

  To see this, consider $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. For this base, we have $d_{\boldsymbol{\beta}^{(0)}}(1) = 2010^\omega$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega$. However, $\frac{5+\sqrt{13}}{6}$ is not an algebraic integer.
- For the same non Pisot algebraic integer δ, there may exist a Parry alternate base $\boldsymbol{\alpha} = (\alpha_0, \cdots, \alpha_{p-1})$ and a non-Parry alternate base $\boldsymbol{\beta} = (\beta_0 \cdots \beta_{p-1})$ such that $\prod_{i=0}^{p-1} \alpha_i = \prod_{i=0}^{p-1} \beta_i = \delta$.

Generalization of Schmidt's results

Define $\mathrm{Per}(\boldsymbol{\beta}) = \{x \in [0,1) : d_{\boldsymbol{\beta}}(x) \text{ is ultimately periodic}\}$.

Theorem (Charlier, Cisternino & Kreczman 2022)

1. *If* $\mathbb{Q} \cap [0,1) \subseteq \bigcap_{i=0}^{p-1} \mathrm{Per}(\boldsymbol{\beta}^{(i)})$ *then* $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ *and* δ *is either a Pisot number or a Salem number.*
2. *If* δ *is a Pisot number and* $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ *then* $\mathrm{Per}(\boldsymbol{\beta}) = \mathbb{Q}(\delta) \cap [0,1)$.

Theorem (Charlier, Cisternino & Kreczman 2022)

If δ *is an algebraic integer that is neither a Pisot number nor a Salem number then* $\mathrm{Per}(\boldsymbol{\beta}) \cap \mathbb{Q}$ *is nowhere dense in* $[0,1)$.

Alternate bases whose set of zero representations is accepted by a finite Büchi automaton

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

The following assertions are equivalent.

1. *The set* $Z(\beta, \boldsymbol{D})$ *is accepted by a finite Büchi automaton for all p-tuple of alphabets of integers* $\boldsymbol{D} = (D_0, \dots, D_{p-1})$.
2. *The set* $Z(\beta, \boldsymbol{D})$ *is accepted by a finite Büchi automaton for one p-tuple of alphabets of integers* $\boldsymbol{D} = (D_0, \dots, D_{p-1})$ *such that* $D_i \supseteq \{-\lfloor \beta_i \rfloor, \dots, \lfloor \beta_i \rfloor\}$ *for all* $i \in \{0, \dots, p-1\}$ *and* $\lfloor \beta_j \rfloor \geq \lceil \delta \rceil - 1$ *for some* $j \in \{0, \dots, p-1\}$.
3. δ *is a Pisot number and* $\beta_0, \dots, \beta_{p-1} \in \mathbb{Q}(\delta)$.

Normalization in alternate base

The normalization function is the partial function $\nu_{\beta,\boldsymbol{D}}$ mapping any β-representation $a \in \prod_{n\in\mathbb{N}} D_n$ of a real number $x \in [0,1)$ to the β-expansion of x.

We say that $\nu_{\beta,\boldsymbol{D}}$ is computable by a finite Büchi automaton if there exists a finite Büchi automaton accepting the set

$$\left\{(u,v) \in \prod_{n\in\mathbb{N}}(D_n \times \{0,\ldots,\lceil\beta_n\rceil - 1\}) : \mathrm{val}_{\beta}(u) = \mathrm{val}_{\beta}(v) \text{ and } \exists x \in [0,1),\ v = d_{\beta}(x)\right\}.$$

First ingredient.

Consider two p-tuples of alphabets $\boldsymbol{D} = (D_0,\ldots,D_{p-1})$ and $\boldsymbol{D'} = (D'_0,\ldots,D'_{p-1})$.
We set $\boldsymbol{D} - \boldsymbol{D'} = (D_0 - D'_0,\ldots,D_{p-1} - D'_{p-1})$.

From the zero automaton $\mathcal{Z}(\beta,\boldsymbol{D}-\boldsymbol{D'})$, we define a converter $\mathcal{C}_{\beta,\boldsymbol{D},\boldsymbol{D'}}$ from $\boldsymbol{D}$ to $\boldsymbol{D'}$, that is, a Büchi automaton accepting the set

$$\{(u,v) \in \prod_{n\in\mathbb{N}}(D_n \times D'_n) : \mathrm{val}_{\beta}(u) = \mathrm{val}_{\beta}(v)\}.$$

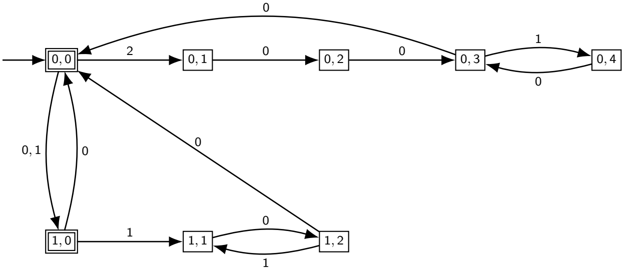
Proposition

If δ is a Pisot number and $\beta_0,\ldots,\beta_{p-1} \in \mathbb{Q}(\delta)$, then the converter $\mathcal{C}_{\beta,\boldsymbol{D},\boldsymbol{D'}}$ is finite.

Second ingredient.

In the case where β is a Parry alternate base, we can define a Büchi automaton accepting the set $\{d_{\boldsymbol{\beta}}(x) : x \in [0,1)\}$.

For $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$, we have seen that $d^*_{\boldsymbol{\beta}^{(0)}}(1) = 200(10)^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$.



Combining these two automata, we obtain the following result.

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$, then the normalization function $\nu_{\boldsymbol{\beta},\boldsymbol{D}}$ is computable by a finite Büchi automaton.

Thank you!