Regular sequences in abstract numeration systems

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Part 1

$(\mathcal{S}, \mathbb{K})$-regular sequences

Abstract numeration systems

An ANS is a triple $\mathcal{S} = (L, A, <)$ where L is an infinite regular language over a totally ordered alphabet $(A, <)$.

The words in L are ordered with respect to the radix order $<_{\mathrm{rad}}$ induced by the order $<$ on A.

The $\mathcal{S}$-representation function $\mathrm{rep}_{\mathcal{S}} \colon \mathbb{N} \to L$ maps any non-negative integer n onto the nth word in L.

The $\mathcal{S}$-value function $\mathrm{val}_{\mathcal{S}} \colon L \to \mathbb{N}$ is the reciprocal function of $\mathrm{rep}_{\mathcal{S}}$.

(Lecomte & Rigo 2001)

Runnning example: $\mathcal{S} = (a^*b^*, a < b)$

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n	$\mathrm{rep}_{\mathcal{S}}(n)$	n	$\mathrm{rep}_{\mathcal{S}}(n)$	n	$\mathrm{rep}_{\mathcal{S}}(n)$
0	ε	8	abb	16	$aaaab$
1	a	9	bbb	17	$aaabb$
2	b	10	$aaaa$	18	$aabbb$
3	aa	11	$aaab$	19	$abbbb$
4	ab	12	$aabb$	20	$bbbbb$
5	bb	13	$abbb$	21	$aaaaaa$
6	aaa	14	$bbbb$	22	$aaaaab$
7	aab	15	$aaaaa$	23	$aaaabb$

$$\mathrm{val}_{\mathcal{S}}(a^p b^q) = \frac{(p+q)(p+q+1)}{2} + q$$

Other examples

- Integer base b numeration systems correspond to the ANS $\mathcal{S}_b = (\{1, \ldots, b-1\}\{0, \ldots, b-1\}^* \cup \{\varepsilon\}, 0 < 1 < \cdots < b-1)$.
- The Zeckendorf numeration system corresponds to the ANS $\mathcal{S}_F = (1\{0, 01\}^* \cup \{\varepsilon\}, 0 < 1)$.
- More generally, numeration systems based on a sequence $U = (U_i)_{i \geq 0}$ and having a regular numeration language.
- All Pisot numeration systems.

Representing elements of $\mathbb{N}^d$

We will work with a d-tuple $\boldsymbol{\mathcal{S}} = (\mathcal{S}_1, \ldots, \mathcal{S}_d)$ of ANS

$$\mathcal{S}_1 = (L_1, A_1, <_1), \ \ldots, \ \mathcal{S}_d = (L_d, A_d, <_d).$$

Let $\# \notin A_1 \cup \cdots \cup A_d$ and the numeration alphabet is

$$\boldsymbol{A} = ((A_1 \cup \{\#\}) \times \cdots \times (A_d \cup \{\#\})) \setminus \left\{ \begin{pmatrix} \# \\ \vdots \\ \# \end{pmatrix} \right\}.$$

For a d-tuple

$$\begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix} \in A_1^* \times \cdots \times A_d^*$$

we set

$$\begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix}^{\#} = \begin{pmatrix} \#^{\ell - |w_1|} w_1 \\ \vdots \\ \#^{\ell - |w_d|} w_d \end{pmatrix} \in \boldsymbol{A}^*$$

where $\ell = \max\{|w_1|, \ldots, |w_d|\}$.

The numeration language is $\boldsymbol{L} = (L_1 \times \cdots \times L_d)^{\#}$.

Since the languages $L_1, \ldots, L_d$ are regular, $\boldsymbol{L}$ is a regular language over $\boldsymbol{A}$.

Then

$$\mathrm{rep}_{\boldsymbol{\mathcal{S}}} \colon \mathbb{N}^d \to \boldsymbol{L},\ \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{rep}_{\mathcal{S}_1}(n_1) \\ \vdots \\ \mathrm{rep}_{\mathcal{S}_d}(n_d) \end{pmatrix}^{\#}$$

and

$$\mathrm{val}_{\boldsymbol{\mathcal{S}}} \colon \boldsymbol{L} \to \mathbb{N}^d,\ \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{val}_{\mathcal{S}_1}(\tau_{\#}(w_1)) \\ \vdots \\ \mathrm{val}_{\mathcal{S}_d}(\tau_{\#}(w_d)) \end{pmatrix}$$

where $\tau_{\#}$ is the morphism that erases the letter $\#$ and leaves the other letters unchanged.

Running Example

Consider the 2-dimensional ANS $\boldsymbol{\mathcal{S}} = (\mathcal{S}, \mathcal{S})$.

We have

- $\boldsymbol{A} = \{ \binom{\#}{a}, \binom{\#}{b}, \binom{a}{\#}, \binom{a}{a}, \binom{a}{b}, \binom{b}{\#}, \binom{b}{a}, \binom{b}{b} \}$
- $\boldsymbol{L} = (a^* b^* \times a^* b^*)^{\#}$.

For instance,

- $\mathrm{rep}_{\boldsymbol{\mathcal{S}}} \binom{4}{9} = \binom{\#ab}{bbb} = \binom{\#}{b} \binom{a}{b} \binom{b}{b}$
- $\mathrm{val}_{\boldsymbol{\mathcal{S}}} \binom{aab}{\#\#a} = \begin{pmatrix} \mathrm{val}_{\mathcal{S}}(aab) \\ \mathrm{val}_{\mathcal{S}}(a) \end{pmatrix} = \binom{7}{1}$.

$(\mathcal{S}, \mathbb{K})$-Regular sequences

In this talk, $\mathbb{K}$ designates an arbitrary commutative semiring.

A sequence $f \colon \mathbb{N}^d \to \mathbb{K}$ is called $(\mathcal{S}, \mathbb{K})$-regular if the series

$$S_f := \sum_{\boldsymbol{w} \in \boldsymbol{L}} f(\mathrm{val}_{\mathcal{S}}(\boldsymbol{w}))\, \boldsymbol{w}$$

is $\mathbb{K}$-recognizable.

Background on noncommutative formal series

A series is an application

$$S\colon A^* \to \mathbb{K},\ w \mapsto (S, w).$$

It is also denoted

$$\sum_{w \in A^*} (S, w) w.$$

A series is $\mathbb{K}$-recognizable if there exist $\mu\colon A \to \mathbb{K}^{r \times r}$, $\lambda \in \mathbb{K}^{1 \times r}$ and $\gamma \in \mathbb{K}^{r \times 1}$ such that

$$\forall a_1, \ldots, a_\ell \in A, \quad (S, a_1 \cdots a_\ell) = \lambda \mu(a_1) \cdots \mu(a_\ell) \gamma.$$

The triple (λ, μ, γ) is called a linear representation of S.

Running Example

Consider the sequence

$$f\colon \mathbb{N}^2 \to \mathbb{N},\ \left(\begin{smallmatrix} m \\ n \end{smallmatrix}\right) \mapsto \max \left|\mathrm{Suff}(\mathrm{rep}_{\mathcal{S}}(m)) \cap \mathrm{Suff}(\mathrm{rep}_{\mathcal{S}}(n))\right|.$$

We have

$$S_f = \sum_{\boldsymbol{w} \in \boldsymbol{L}} (S, \boldsymbol{w})\boldsymbol{w}$$

where

$$S\colon \boldsymbol{A}^* \to \mathbb{N},\ \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \mapsto \max \left|\mathrm{Suff}(u) \cap \mathrm{Suff}(v)\right|.$$

Since $\boldsymbol{L}$ is a regular language, the series S_f is $\mathbb{N}$-recognizable if so is S.

A linear representation (λ, μ, γ) of S is given by

$$\lambda = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right),\ \gamma = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right),$$

$$\mu\left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right) = \mu\left(\begin{smallmatrix} b \\ b \end{smallmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\mu(\boldsymbol{a}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } \boldsymbol{a} \in \boldsymbol{A} \setminus \left\{\left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ b \end{smallmatrix}\right)\right\}.$$

Thus, S_f is $\mathbb{N}$-recognizable, and hence the sequence f is $(\boldsymbol{\mathcal{S}}, \mathbb{N})$-regular.

Unidimensional versus multidimensional

If we take the convention to pad representations on the right, then we get a different notion of $(\mathcal{S}, \mathbb{K})$-regular sequences.

In the unidimensional case, the two notions coincide (since no padding is necessary).

However, there is no such nice analogy in higher dimensions since it might be that a left $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular sequence is not a right $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular sequence, or vice-versa.

Part 2

$\mathcal{S}$-kernel of a sequence

$\mathcal{S}$-kernel of a sequence

Working hypothesis (WH). The numeration language $\boldsymbol{L}$ is prefix-closed:

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{A}^*, \quad \boldsymbol{uv} \in \boldsymbol{L} \implies \boldsymbol{u} \in \boldsymbol{L}.$$

(This amounts to asking that all languages $L_1, \ldots, L_d$ are prefix-closed.)

For $f \colon \mathbb{N}^d \to \mathbb{K}$ and $\boldsymbol{w} \in \boldsymbol{A}^*$, we define a sequence

$$f \circ \boldsymbol{w} \colon \mathbb{N}^d \to \mathbb{K}$$

by setting

$$\forall \boldsymbol{n} \in \mathbb{N}^d, \quad (f \circ \boldsymbol{w})(\boldsymbol{n}) = \begin{cases} f(\mathrm{val}_{\mathcal{S}}(\mathrm{rep}_{\mathcal{S}}(\boldsymbol{n})\boldsymbol{w})) & \text{if } \mathrm{rep}_{\mathcal{S}}(\boldsymbol{n})\boldsymbol{w} \in \boldsymbol{L} \\ 0 & \text{else.} \end{cases}$$

The $\mathcal{S}$-kernel of f is the set $\ker_{\mathcal{S}}(f) = \{f \circ \boldsymbol{w} \colon \boldsymbol{w} \in \boldsymbol{A}^*\}$.

These definitions generalize those of Berstel & Reutenauer 2011.

Running Example

For all $w \in a^* b^*$, we have $wb \in a^* b^*$ so we get that

$$\forall \boldsymbol{n} \in \mathbb{N}^2, \quad (f \circ \binom{b}{b})(\boldsymbol{n}) = f(\boldsymbol{n}) + 1.$$

Some values of the function $f \circ \binom{ab}{ab}$:

$\boldsymbol{n}$	$\binom{0}{1}$	$\binom{1}{2}$	$\binom{3}{2}$	$\binom{6}{3}$
$\mathrm{rep}_{\mathcal{S}}(\boldsymbol{n})$	$\binom{\#}{a}$	$\binom{a}{b}$	$\binom{aa}{\#b}$	$\binom{aaa}{\#aa}$
$\mathrm{rep}_{\mathcal{S}}(\boldsymbol{n}) \binom{ab}{ab}$	$\binom{\#ab}{aab}$	$\binom{aab}{bab}$	$\binom{aaab}{\#bab}$	$\binom{aaaab}{\#aaab}$
$\mathrm{val}_{\mathcal{S}}(\mathrm{rep}_{\mathcal{S}}(\boldsymbol{n}) \binom{ab}{ab})$	$\binom{4}{7}$	$\nexists$	$\nexists$	$\binom{16}{11}$
$(f \circ \binom{ab}{ab})(\boldsymbol{n})$	2	0	0	4

Left-right duality

Like for $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular sequences, the notion of $\boldsymbol{\mathcal{S}}$-kernel is not left-right symmetric.

The $\boldsymbol{\mathcal{S}}$-kernel defined above may be seen as the right $\boldsymbol{\mathcal{S}}$-kernel.

The left $\boldsymbol{\mathcal{S}}$-kernel of a sequence $f \colon \mathbb{N}^d \to \mathbb{K}$ would then be the set of sequences $\{\boldsymbol{w} \circ f \colon \boldsymbol{w} \in \boldsymbol{A}^*\}$ where

$$\forall \boldsymbol{n} \in \mathbb{N}^d, \quad (\boldsymbol{w} \circ f)(\boldsymbol{n}) = \begin{cases} f(\mathrm{val}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{w}\,\mathrm{rep}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{n}))) & \text{if } \boldsymbol{w}\,\mathrm{rep}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{n}) \in \boldsymbol{L} \\ 0 & \text{else.} \end{cases}$$

In this case, we need to adapt the conventions used so far:

- We pad representations of vectors of integers with $\#$'s on the right.
- We ask the numeration language $\boldsymbol{L}$ to be suffix-closed.

Provided that these conventions are taken, all our results can be adapted to the left version of the $\boldsymbol{\mathcal{S}}$-kernel and left $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular sequences.

First characterization of $(\mathcal{S}, \mathbb{K})$-regular sequences

A $\mathbb{K}$-submodule of $\mathbb{K}^{\mathbb{N}^d}$ is called stable if it is closed under all operations

$$\mathbb{K}^{\mathbb{N}^d} \to \mathbb{K}^{\mathbb{N}^d},\ f \mapsto f \circ \boldsymbol{w}$$

for all $\boldsymbol{w} \in \boldsymbol{A}^*$.

Theorem (Charlier-Cisternino-Stipulanti 2020)

A sequence $f : \mathbb{N}^d \to \mathbb{K}$ is $(\mathcal{S}, \mathbb{K})$-regular if and only if there exists a stable finitely generated $\mathbb{K}$-submodule of $\mathbb{K}^{\mathbb{N}^d}$ containing f.

The proof of this result generalizes ideas from Berstel & Reutenauer 2011.

It relies on the property (under WH) that for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{A}^*$ and $f : \mathbb{N}^d \to \mathbb{K}$, $(f \circ \boldsymbol{v}) \circ \boldsymbol{u} = f \circ \boldsymbol{uv}$.

Remark: The latter property cannot be obtained from the notion of $\mathcal{S}$-kernel used in Rigo & Maes 2002.

Second characterization of $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular sequences

The following result is a practical criterion for $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regularity.

Theorem (Charlier-Cisternino-Stipulanti 2020)

A sequence $f : \mathbb{N}^d \to \mathbb{K}$ is $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular if and only if there exist $r \in \mathbb{N}$ and $f_1, f_2, \ldots, f_r : \mathbb{N}^d \to \mathbb{K}$ such that $f = f_1$ and for all $\boldsymbol{a} \in \boldsymbol{A}$ and all $i \in [\![1, r]\!]$, there exist $k_{\boldsymbol{a},i,1}, \ldots, k_{\boldsymbol{a},i,r} \in \mathbb{K}$ such that

$$f_i \circ \boldsymbol{a} = \sum_{j=1}^{r} k_{\boldsymbol{a},i,j} f_j.$$

Running Example

Define the sequence

$$g\colon \mathbb{N}^2 \to \mathbb{K},\ \boldsymbol{n} \mapsto \begin{cases} f(\boldsymbol{n}) & \text{if } \boldsymbol{n} \in \mathrm{val}_{\mathcal{S}}(a^*) \times \mathrm{val}_{\mathcal{S}}(a^*) \\ 0 & \text{else.} \end{cases}$$

The sequences

1. f
2. g
3. $\chi_{\{0\}\times\mathrm{val}_{\mathcal{S}}(a^*)}$
4. $\chi_{\{0\}\times\mathbb{N}}$
5. $\chi_{\mathrm{val}_{\mathcal{S}}(a^*)\times\{0\}}$
6. $\chi_{\mathrm{val}_{\mathcal{S}}(a^*)\times\mathrm{val}_{\mathcal{S}}(a^*)}$
7. $\chi_{\mathrm{val}_{\mathcal{S}}(a^*)\times\mathbb{N}}$
8. $\chi_{\mathbb{N}\times\{0\}}$
9. $\chi_{\mathbb{N}\times\mathrm{val}_{\mathcal{S}}(a^*)}$
10. 1

satisfy the second characterization.

Let $\boldsymbol{a} \in \boldsymbol{A}$. Then

$$f \circ \boldsymbol{a} = \begin{cases} g + \chi_{\mathrm{val}_{\mathcal{S}}(a^*) \times \mathrm{val}_{\mathcal{S}}(a^*)} & \text{if } \boldsymbol{a} = \binom{a}{a} \\ f + 1 & \text{if } \boldsymbol{a} = \binom{b}{b} \\ 0 & \text{else} \end{cases}$$

$$g \circ \boldsymbol{a} = \begin{cases} g + \chi_{\mathrm{val}_{\mathcal{S}}(a^*) \times \mathrm{val}_{\mathcal{S}}(a^*)} & \text{if } \boldsymbol{a} = \binom{a}{a} \\ 0 & \text{else.} \end{cases}$$

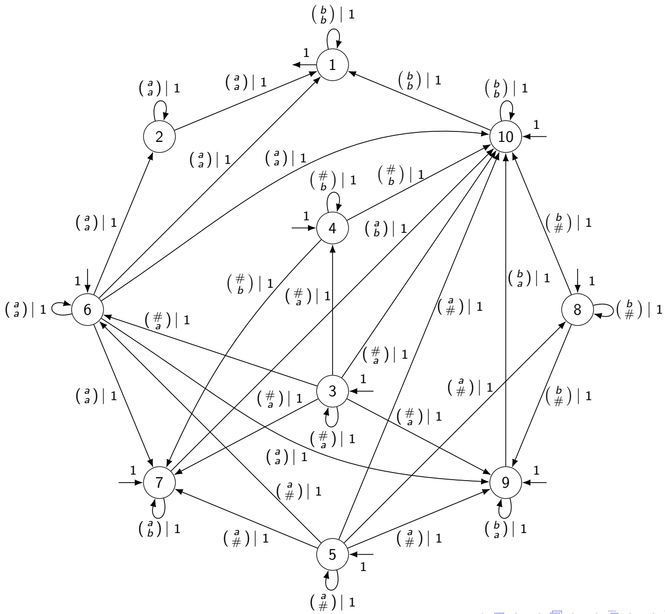
Next, take $X_1, X_2 \in \{\{0\}, \mathrm{val}_{\mathcal{S}}(a^*), \mathbb{N}\}$ such that not both X_1, X_2 are equal to $\{0\}$. Then

$$\chi_{X_1 \times X_2} \circ \boldsymbol{a} = \chi_{Y_1 \times Y_2}$$

where

$$\forall i \in \{1, 2\}, \quad Y_i = \begin{cases} \{0\} & \text{if } a_i = \# \\ \mathrm{val}_{\mathcal{S}}(a^*) & \text{if } a_i = a \text{ and } X_i \in \{\mathrm{val}_{\mathcal{S}}(a^*), \mathbb{N}\} \\ \mathbb{N} & \text{if } a_i = b \text{ and } X_i = \mathbb{N} \\ \emptyset & \text{else.} \end{cases}$$

An $\mathbb{N}$-automaton recognizing the series S_f



Third characterization (whenever $\mathbb{K}$ is finite or is a ring)

Since $\langle \ker_{\mathcal{S}}(f) \rangle_{\mathbb{K}}$ is stable, it is the smallest stable $\mathbb{K}$-submodule of $\mathbb{K}^{\mathbb{N}^d}$ containing f.

For an arbitrary commutative semiring $\mathbb{K}$, the fact that f is a $(\mathcal{S}, \mathbb{K})$-regular sequence does not imply that $\langle \ker_{\mathcal{S}}(f) \rangle_{\mathbb{K}}$ is finitely generated.

The following theorem provides us with some cases where $\langle \ker_{\mathcal{S}}(f) \rangle_{\mathbb{K}}$ is indeed finitely generated.

Theorem (Charlier-Cisternino-Stipulanti 2020)

Let $f \colon \mathbb{N}^d \to \mathbb{K}$ be a sequence.

- *If $\langle \ker_{\mathcal{S}}(f) \rangle_{\mathbb{K}}$ is finitely generated then f is $(\mathcal{S}, \mathbb{K})$-regular.*
- *If f is $(\mathcal{S}, \mathbb{K})$-regular and if moreover $\mathbb{K}$ is finite or is a ring, then $\langle \ker_{\mathcal{S}}(f) \rangle_{\mathbb{K}}$ is finitely generated.*

Running Example

The kernel $\langle \ker_{\boldsymbol{\mathcal{S}}}(f) \rangle_{\mathbb{N}}$ is not finitely generated.

- For $\boldsymbol{w} \in \boldsymbol{A}^* \setminus \binom{a}{a}^* \binom{b}{b}^*$, we have $f \circ \boldsymbol{w} = 0$.
- For $k \in \mathbb{N}$, we have $f \circ \binom{b}{b}^k = f + k$.
- For $k, k' \in \mathbb{N}$ with $k \geq 1$, we have

$$\left(f \circ \binom{a}{a}^k \binom{b}{b}^{k'} \right)(\boldsymbol{n}) = \begin{cases} f(\boldsymbol{n}) + k + k' & \text{if } \boldsymbol{n} \in \left(\mathrm{val}_{\mathcal{S}}(a^*)\right)^2 \\ 0 & \text{else.} \end{cases}$$

However, since f is $(\boldsymbol{\mathcal{S}}, \mathbb{N})$-regular, hence also $(\boldsymbol{\mathcal{S}}, \mathbb{Z})$-regular, our third characterization implies that $\langle \ker_{\boldsymbol{\mathcal{S}}}(f) \rangle_{\mathbb{Z}}$ is finitely generated.

Indeed, it is easily seen that

$$\langle \ker_{\boldsymbol{\mathcal{S}}}(f) \rangle_{\mathbb{Z}} = \langle f, f \circ \binom{a}{a}, f \circ \binom{b}{b}, f \circ \binom{aa}{aa} \rangle_{\mathbb{Z}}.$$

Part 3

$\mathcal{S}$-Automatic sequences

Characterization of $\mathcal{S}$-automatic sequences

A sequence $f\colon \mathbb{N}^d \to \Delta$ is called $\mathcal{S}$-automatic if there exists a DFAO $\mathcal{A} = (Q, q_0, \delta, \boldsymbol{A}, \tau, \Delta)$ such that

$$\forall \boldsymbol{n} \in \mathbb{N}^d, \quad f(\boldsymbol{n}) = \tau(\delta(q_0, \mathrm{rep}_{\mathcal{S}}(\boldsymbol{n}))).$$

This definition was introduced in Rigo 2000.

In this work, we consider sequences f with images in $\mathbb{K}$, so the output alphabet Δ is seen as a subset of $\mathbb{K}$.

Theorem (Charlier-Cisternino-Stipulanti 2020)

A sequence $f\colon \mathbb{N}^d \to \mathbb{K}$ is $\mathcal{S}$-automatic if and only if $\ker_{\mathcal{S}}(f)$ is finite.

Even though, for $d = 1$, the statement of this result coincide with that of a result from Rigo & Maes 2002, this is indeed a new result since we are working with a different notion of $\mathcal{S}$-kernel.

As a consequence, we obtain that, for any given sequence f, both kernels are simultaneously finite.

Characterization of $\boldsymbol{\mathcal{S}}$-automatic sequences among $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular sequences (whenever $\mathbb{K}$ is finite or is a ring)

Theorem (Charlier-Cisternino-Stipulanti 2020)

Let $f : \mathbb{N}^d \to \mathbb{K}$.

- *If f is $\boldsymbol{\mathcal{S}}$-automatic then it is $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular.*
- *If f is $(\boldsymbol{\mathcal{S}}, \mathbb{K})$-regular and takes only finitely many values, and if moreover $\mathbb{K}$ is finite or is a ring, then f is $\boldsymbol{\mathcal{S}}$-automatic.*

Part 4

Enumerating $\mathcal{S}$-recognizable properties of $\mathcal{S}$-automatic sequences give rise to $(\mathcal{S}, \mathbb{N})$-regular sequences

First ingredient: generating $(\boldsymbol{\mathcal{S}}, \mathbb{N})$-regular sequences from $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')$-recognizable sets

In this part, we focus on the semirings $\mathbb{N}$ and $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$.

A subset X of $\mathbb{N}^d$ is $\boldsymbol{\mathcal{S}}$-recognizable if the language $\mathrm{rep}_{\boldsymbol{\mathcal{S}}}(X)$ is regular.

Ingredient 1 (Charlier-Cisternino-Stipulanti 2020)

Let $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}'$ be d- and d'-dimensional ANS respectively.
If X is an $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')$-recognizable subset of $\mathbb{N}^{d+d'}$, then the sequence

$$f \colon \mathbb{N}^d \to \mathbb{N}_\infty, \; \boldsymbol{n} \mapsto \mathrm{Card}\{\boldsymbol{n}' \in \mathbb{N}^{d'} : \left(\begin{smallmatrix} \boldsymbol{n} \\ \boldsymbol{n}' \end{smallmatrix}\right) \in X\}$$

is $(\boldsymbol{\mathcal{S}}, \mathbb{N}_\infty)$-regular. If moreover $f(\mathbb{N}) \subseteq \mathbb{N}$ then f is $(\boldsymbol{\mathcal{S}}, \mathbb{N})$-regular.

Second ingredient: $\boldsymbol{\mathcal{S}}$-recognizable enumerations of $\mathbb{N}^d$

We define an enumeration $E_{\boldsymbol{\mathcal{S}}} \colon \mathbb{N}^d \to \mathbb{N}$ recursively as follows.

We fix a total order on $\boldsymbol{A}$ and we consider the induced radix order on $\boldsymbol{A}^*$.

Then we define a total order $<_{\boldsymbol{\mathcal{S}}}$ on $\mathbb{N}^d$ by declaring that

$$\forall \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^d, \quad \boldsymbol{m} <_{\boldsymbol{\mathcal{S}}} \boldsymbol{n} \iff \mathrm{rep}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{m}) <_{\mathrm{rad}} \mathrm{rep}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{n}).$$

For all $\boldsymbol{n} \in \mathbb{N}^d$, we define

$$E_{\boldsymbol{\mathcal{S}}}(\boldsymbol{n}) = i$$

if $\boldsymbol{n}$ is the i-th element of $\mathbb{N}^d$ w.r.t. $<_{\boldsymbol{\mathcal{S}}}$.

Ingredient 2

For each $\diamond \in \{=, >, <\}$, the set

$$\{\left(\begin{smallmatrix}\boldsymbol{m}\\ \boldsymbol{n}\end{smallmatrix}\right) \in \mathbb{N}^{2d} \colon E_{\boldsymbol{\mathcal{S}}}(\boldsymbol{m}) \diamond E_{\boldsymbol{\mathcal{S}}}(\boldsymbol{n})\}$$

is $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}})$-recognizable.

Running Example

Assuming $\binom{\#}{a} < \binom{\#}{b} < \binom{a}{\#} < \binom{a}{a} < \binom{a}{b} < \binom{b}{\#} < \binom{b}{a} < \binom{b}{b}$, we obtain

	ε	a	b	aa	ab	bb
bb	15	16	17	28	29	35
ab	10	12	14	25	27	34
aa	9	11	13	24	26	33
b	2	5	8	20	23	32
a	1	4	7	19	22	31
ε	0	3	6	18	21	30

Third ingredient: $\boldsymbol{\mathcal{S}}$-recognizable predicates

A predicate P on $\mathbb{N}^{md}$ is $\boldsymbol{\mathcal{S}}$-recognizable is the set

$$\{\boldsymbol{n} \in \mathbb{N}^{md} : P(\boldsymbol{n}) \text{ is true}\}$$

is $(\boldsymbol{\mathcal{S}}, \dots, \boldsymbol{\mathcal{S}})$-recognizable (where $\boldsymbol{\mathcal{S}}$ is repeated m times).

The following result generalizes ideas from Bruyère, Hansel, Michaux & Villemaire 1996 and Charlier, Rampersad & Shallit 2012 to ANS.

Ingredient 3

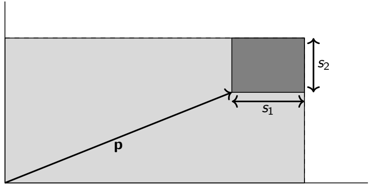
Any predicate on $\mathbb{N}^{md}$ that is defined recursively from $\boldsymbol{\mathcal{S}}$-recognizable predicates by only using the logical connectives $\wedge, \vee, \neg, \implies, \iff$ and the quantifiers $\forall$ and $\exists$ on variables describing elements of $\mathbb{N}^d$, is $\boldsymbol{\mathcal{S}}$-recognizable.

Corollary

If P a such a predicate on $\mathbb{N}^d$ then the closed predicates $\forall \boldsymbol{x} P(\boldsymbol{x})$, $\exists \boldsymbol{x} P(\boldsymbol{x})$ and $\exists^\infty \boldsymbol{x} P(\boldsymbol{x})$ are decidable.

Application to factor complexity

The factor complexity of $f\colon \mathbb{N}^d \to \mathbb{K}$ is the function $\rho_f\colon \mathbb{N}^d \mapsto \mathbb{N}_\infty$ that maps each $\boldsymbol{s} \in \mathbb{N}^d$ to the number of factors of size $\boldsymbol{s}$ occurring in f.



If the sequence f has a finite image (as is the case for automatic sequences) then for all $\boldsymbol{s} \in \mathbb{N}^d$, $\rho_f(\boldsymbol{s}) \in \mathbb{N}$.

Theorem (Charlier-Cisternino-Stipulanti 2020)

Let $\mathcal{S}$ be an ANS such that addition is $\mathcal{S}$-recognizable, i.e., the $3d$-ary predicate $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$ is $\mathcal{S}$-recognizable. Then the factor complexity of an $\mathcal{S}$-automatic sequence is an $(\mathcal{S}, \mathbb{N})$-regular sequence.

Proof.

Let f be an $\mathcal{S}$-automatic d-dimensional sequence.

For all $\boldsymbol{s} \in \mathbb{N}^d$, $\rho_f(\boldsymbol{s})$ is equal to

$$\mathrm{Card}\{\boldsymbol{p} \in \mathbb{N}^d : \forall \boldsymbol{p}' \in \mathbb{N}^d \left(E_{\mathcal{S}}(\boldsymbol{p}') < E_{\mathcal{S}}(\boldsymbol{p}) \implies \exists \boldsymbol{i} < \boldsymbol{s},\ f(\boldsymbol{p}' + \boldsymbol{i}) \neq f(\boldsymbol{p} + \boldsymbol{i})\right)\}.$$

By Ingredient 1, it suffices to prove that the set

$$X := \{(\boldsymbol{s}, \boldsymbol{p}) \in \mathbb{N}^{2d} : \forall \boldsymbol{p}' \in \mathbb{N}^d \left(E_{\mathcal{S}}(\boldsymbol{p}') < E_{\mathcal{S}}(\boldsymbol{p}) \implies \exists \boldsymbol{i} < \boldsymbol{s},\ f(\boldsymbol{p}' + \boldsymbol{i}) \neq f(\boldsymbol{p} + \boldsymbol{i})\right)\}$$

is $(\mathcal{S}, \mathcal{S})$-recognizable.

By Ingredient 2, the predicate $E_{\mathcal{S}}(\boldsymbol{p}') < E_{\mathcal{S}}(\boldsymbol{p})$ is $\mathcal{S}$-recognizable.

By Ingredient 3, since f is $\mathcal{S}$-automatic and addition is $\mathcal{S}$-recognizable, we get that X is $\mathcal{S}$-recognizable. □

Part 5

$(\mathcal{S}, \mathcal{S}')$-Synchronized sequences

$(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')$-Synchronized sequences

In this section, we consider a $(d + d')$-dimensional ANS $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}') = (\mathcal{S}_1, \ldots, \mathcal{S}_d, \mathcal{S}'_1, \ldots, \mathcal{S}'_{d'})$.

A sequence $f\colon \mathbb{N}^d \to \mathbb{N}^{d'}$ is $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')$-synchronized if its graph

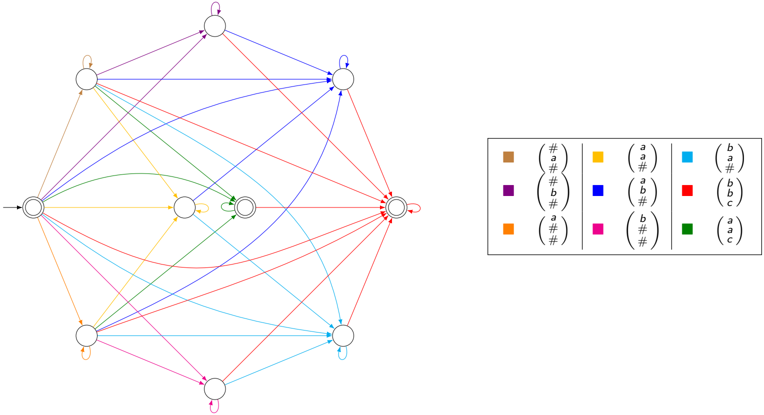
$$\mathcal{G}_f = \{\left(\begin{smallmatrix} \boldsymbol{n} \\ f(\boldsymbol{n}) \end{smallmatrix} \right) : \boldsymbol{n} \in \mathbb{N}^d\}$$

is an $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')$-recognizable subset of $\mathbb{N}^{d+d'}$.

Running Example

By a pumping argument, it is easily seen that the sequence f is not $(\boldsymbol{\mathcal{S}}, \mathcal{S})$-synchronized.

However, the sequence f is $(\boldsymbol{\mathcal{S}}, \mathcal{S}_c)$-synchronized where $\mathcal{S}_c = (c^*, c)$.



Proposition

- *For all* $\boldsymbol{k} \in \mathbb{N}^d$*, the sequence* $\mathbb{N}^d \to \mathbb{N}^d,\ \boldsymbol{n} \mapsto \boldsymbol{n}+\boldsymbol{k}$ *is* $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}})$*-synchronized.*
- *Any* $\boldsymbol{\mathcal{S}}$*-automatic sequence* $f\colon \mathbb{N}^d \to \mathbb{N}^{d'}$ *is* $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')$*-synchronized.*
- *Any* $(\boldsymbol{\mathcal{S}}, \mathcal{S}')$*-synchronized sequence* $f\colon \mathbb{N}^d \to \mathbb{N}$ *is* $(\boldsymbol{\mathcal{S}}, \mathbb{N})$*-regular.*

Sketch of the proof

The 1st item generalizes a result from Charlier, Lacroix & Rampersad 2011.

The 3rd item follows from Ingredient 3 since for any $f\colon \mathbb{N}^d \to \mathbb{N}$, we have

$$\begin{aligned}\forall \boldsymbol{n} \in \mathbb{N}^d, \quad f(\boldsymbol{n}) &= \mathrm{Card}\{\ell \in \mathbb{N} \colon \ell < f(\boldsymbol{n})\} \\ &= \mathrm{Card}\{\ell \in \mathbb{N} \colon \exists m,\ \left(\begin{smallmatrix}\boldsymbol{n}\\ m\end{smallmatrix}\right) \in \mathcal{G}_f \ \wedge\ \ell < m\}.\end{aligned}$$

Even though both families of $\boldsymbol{\mathcal{S}}$-automatic sequences and $(\boldsymbol{\mathcal{S}}, \mathbb{N})$-regular sequences are closed under sum, product and product by a constant, it is no longer the case of the family of $(\boldsymbol{\mathcal{S}}, \mathcal{S}')$-synchronized sequences.

For instance, the sequence $\mathbb{N} \to \mathbb{N},\ n \mapsto n$ is $(\mathcal{S}, \mathcal{S})$-synchronized for any abstract numeration system $\mathcal{S}$.

However, the sequence $\mathbb{N} \to \mathbb{N},\ n \mapsto 2n$ is not $(\mathcal{S}, \mathcal{S})$-synchronized in general.

For example, it is not for the unary system $\mathcal{S} = (c^*, c)$ since the language

$$\left\{ \begin{pmatrix} \#^n c^n \\ c^{2n} \end{pmatrix} : n \in \mathbb{N} \right\}$$

is not regular.

Synchronized relations

The graph of a relation $R\colon A^* \to B^*$ (where A and B are arbitrary alphabets) is

$$\mathcal{G}_R = \{\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in A^* \times B^* \colon uRv\}.$$

Let $\$ \notin A \cup B$. A relation $R\colon A^* \to B^*$ is synchronized if the language

$$(\mathcal{G}_R)^{\$} = \{\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)^{\$} \colon \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathcal{G}_R\}$$

is regular.

For a sequence $f\colon \mathbb{N}^d \to \mathbb{N}^{d'}$, we define a relation $R_{f,\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}'}}\colon \boldsymbol{A}^* \to (\boldsymbol{A}')^*$ by

$$\mathcal{G}_{R_{f,\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}'}}} = \{\left(\begin{smallmatrix} \boldsymbol{w} \\ \boldsymbol{w}' \end{smallmatrix}\right) \in \boldsymbol{L} \times \boldsymbol{L}' \colon f(\mathrm{val}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{w})) = \mathrm{val}_{\boldsymbol{\mathcal{S}'}}(\boldsymbol{w}')\}.$$

Proposition

A sequence $f\colon \mathbb{N}^d \to \mathbb{N}^{d'}$ is $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}'})$-synchronized if and only if the relation $R_{f,\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}'}}$ is synchronized.

The composition of synchronized sequences is synchronized.

Since the composition of synchronized relations is synchronized (Frougny & Sakarovitch 2010), we obtain the following result.

Proposition

Let $f : \mathbb{N}^d \to \mathbb{N}^{d'}$ *be an* $(\mathcal{S}, \mathcal{S}')$*-synchronized sequence and* $g : \mathbb{N}^{d'} \to \mathbb{N}^{d''}$ *be an* $(\mathcal{S}', \mathcal{S}'')$*-synchronized sequence. Then* $g \circ f : \mathbb{N}^d \to \mathbb{N}^{d''}$ *is* $(\mathcal{S}, \mathcal{S}'')$*-synchronized.*

This contrasts with the well-known fact that the family of regular sequences is not closed under composition (Allouche & Shallit 1992).

Part 6

Mixing regular sequences and synchronized sequences

The composition of a synchronized relation and a $\mathbb{K}$-recognizable series is $\mathbb{K}$-recognizable.

For a relation $R\colon A^* \to B^*$ and a series $S\colon B^* \to \mathbb{K}$ such that for all $u \in A^*$, the language $\{v \in B^* \colon uRv\}$ is finite, we define

$$S \circ R\colon A^* \to \mathbb{K},\ u \mapsto \sum_{\substack{v \in B^* \\ uRv}} (S, v).$$

Theorem (Charlier-Cisternino-Stipulanti 2020)

If R is synchronized and S is $\mathbb{K}$-recognizable, then $S \circ R$ is $\mathbb{K}$-recognizable.

The composition of a synchronized sequence and a regular one is regular.

Corollary

If $f : \mathbb{N}^d \to \mathbb{N}^{d'}$ *is* $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')$*-synchronized and* $g : \mathbb{N}^{d'} \to \mathbb{K}$ *is* $(\boldsymbol{\mathcal{S}}', \mathbb{K})$*-regular, then* $g \circ f : \mathbb{N}^d \to \mathbb{K}$ *is* $(\boldsymbol{\mathcal{S}}, \mathbb{K})$*-regular.*

Corollary

Suppose that $\mathbb{K}$ *is finite or is a ring. If* $f : \mathbb{N}^d \to \mathbb{N}^{d'}$ *is an* $\boldsymbol{\mathcal{S}}$*-automatic sequence and* $g : \mathbb{N}^{d'} \to \mathbb{K}$ *is an* $(\boldsymbol{\mathcal{S}}', \mathbb{K})$*-regular sequence, then the sequence* $g \circ f : \mathbb{N}^d \to \mathbb{K}$ *is* $\boldsymbol{\mathcal{S}}$*-automatic.*

Part 7

Robustness of $(\boldsymbol{U}, \mathbb{K})$-regular sequences for Pisot numeration systems $\boldsymbol{U}$

Theorem (Charlier-Cisternino-Stipulanti 2020)

For $f\colon \mathbb{N}^d \to \mathbb{K}$ and a d-tuple $\boldsymbol{U} = (U_1, \ldots, U_d)$ of Pisot numeration systems, the following assertions are equivalent.

1. *The sequence f is $(\boldsymbol{U}, \mathbb{K})$-regular.*
2. *For all finite alphabets $\boldsymbol{A} \subset \mathbb{Z}^d$, the series $\sum_{\boldsymbol{w} \in \boldsymbol{A}^*} f(||\mathrm{val}_{\boldsymbol{U}}(\boldsymbol{w})||)\, \boldsymbol{w}$ is $\mathbb{K}$-recognizable.*
3. *The series $\sum_{\boldsymbol{w} \in \boldsymbol{A}_{\boldsymbol{U}}^*} f(\mathrm{val}_{\boldsymbol{U}}(\boldsymbol{w}))\, \boldsymbol{w}$ is $\mathbb{K}$-recognizable.*
4. *There exists a $\mathbb{K}$-recognizable series $S\colon \boldsymbol{A}_{\boldsymbol{U}}^* \to \mathbb{K}$ such that for all $\boldsymbol{n} \in \mathbb{N}^d$, $(S, \mathrm{rep}_{\boldsymbol{U}}(\boldsymbol{n})) = f(\boldsymbol{n})$.*

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