Permutations and shifts: a survey

Émilie Charlier

Département de Mathématique, Université de Liège

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Permutations

We write

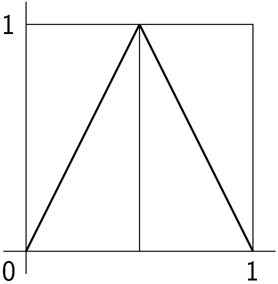
$$312 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

and

$$(312) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = 231.$$

Let's start with the tent map

$$T\colon [0,1] \to [0,1],\ \ x \mapsto \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ -2x+2 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$



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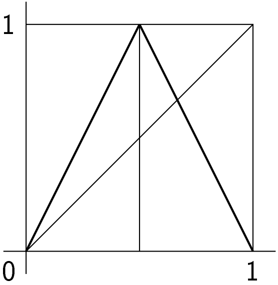

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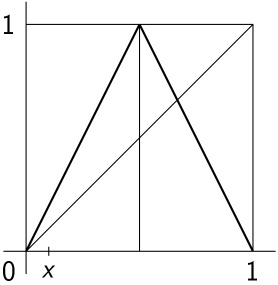

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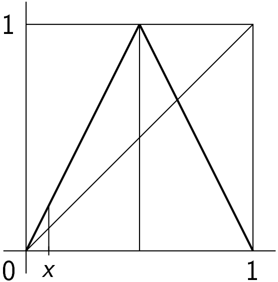

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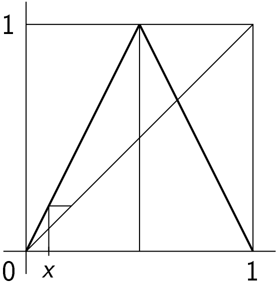

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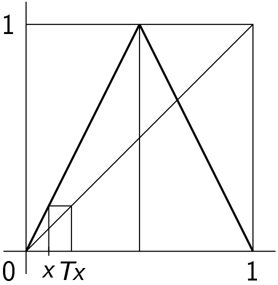

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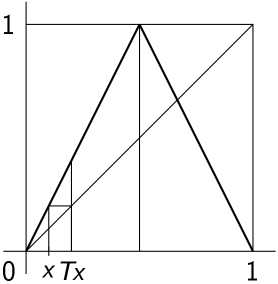

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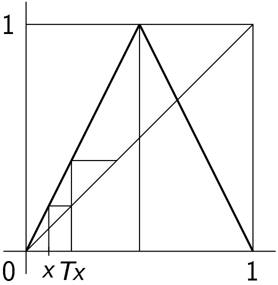

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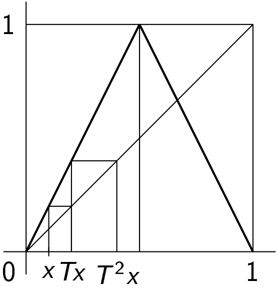

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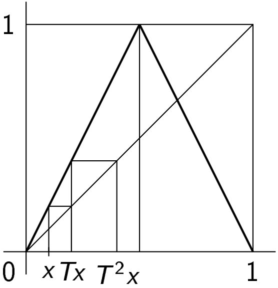

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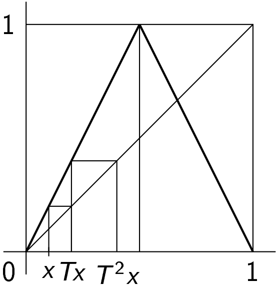

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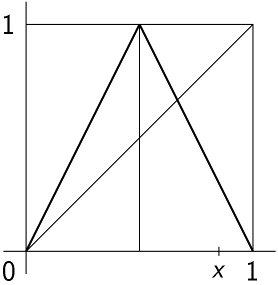

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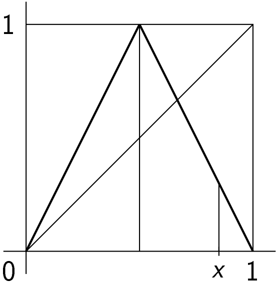

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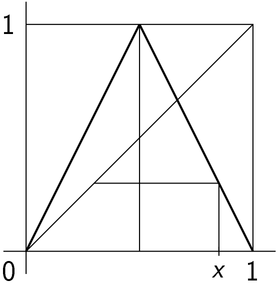

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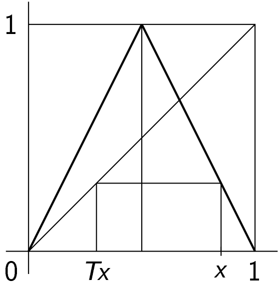

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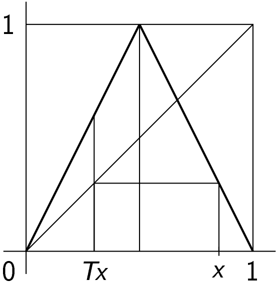

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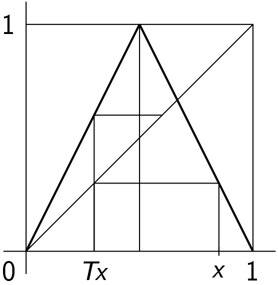

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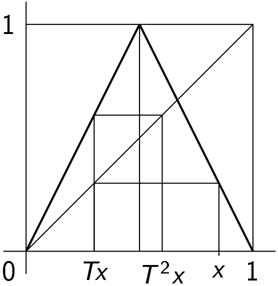

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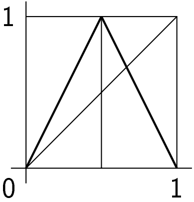
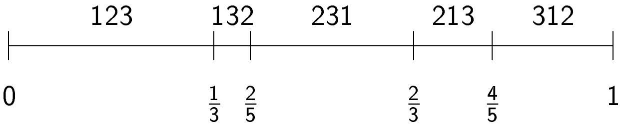


A case study shows



So the permutation $\pi = 321$ is not realizable!

Dynamical systems

- (X, T) where X is a set and T is a map from X to X.
- The objects of study are the trajectories of points.
- The orbit of $x \in X$ is the subset $\{T^n(x) \colon n \in \mathbb{N}\}$.
- Typically, the set X is endowed with a specific structure and the map T preserves this structure.
- If X is a topological space and T is continuous, then (X, T) is a topological dynamical system.
- If X is a measurable space and T is measure preserving, then (X, T) is a measure-preserving dynamical system.

Conjugacy (in the topological case)

- (X_1, T_1) and (X_2, T_2) are conjugate if there exists a homeomorphism $\phi\colon X_1 \to X_2$ such that $\phi \circ T_1 = T_2 \circ \phi$

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ X_2 & \xrightarrow{T_2} & X_2 \end{array}$$

- One of the goals in the theory: classify dynamical systems up to conjugacy.

Invariants

- The idea: if two conjugate systems necessarily share some property, which is called an invariant, then this property can be used to distinguish non-conjugate systems.
- The useful invariants must be computable for a large class of dynamical systems.
- Example of invariant: the number of periodic points.

Entropy

- Permits us to measure the complexity of a dynamical system.
- Invariant under conjugacy.
- Computable for a large variety of dynamical systems.
- So, it is a powerful tool in order to classify dynamical systems.

The starting point

Let I be an interval of $\mathbb{R}$ and consider the dynamical systems (I, T) where $T: I \to I$.

Theorem (Bandt-Keller-Pompe 2002)

- *The concepts of permutation entropy and of topological entropy coincide for piecewise monotone interval maps.*
- *Similar result for the Kolmogorov-Sinai entropy w.r.t. an invariant measure.*

Entropy of interval maps via permutations [Bandt-Keller-Pompe 2002]

Permutation entropy

- Let (X, T) be a dynamical system where X is a totally ordered set.
- For an integer $n \geq 1$ and a point $x \in X$ such that

$$x, T(x), \ldots, T^{n-1}(x)$$

are pairwise distinct, $\mathrm{Pat}(T, n, x)$ denotes the permutation $\pi \in \mathcal{S}_n$ defined by

$$T^{\pi^{-1}(1)-1}(x) < T^{\pi^{-1}(2)-1}(x) < \cdots < T^{\pi^{-1}(n)-1}(x).$$

- Otherwise stated, $\pi(i) < \pi(j)$ for all $i, j \in [\![1, n]\!]$ such that $T^{i-1}(x) < T^{j-1}(x)$.

Example

If $T^3(x) < T(x) < x < T^2(x)$ then $\mathrm{Pat}(T, 4, x) = 3241$.

Permutation entropy

- $\mathrm{Allow}(T, n) = \{\mathsf{Pat}(T, n, x) \colon x \in X\}$ is the set of permutations of length n realized by some $x \in X$.
- $\mathrm{Allow}(T) = \bigcup_{n \geq 1} \mathrm{Allow}(T, n)$.
- The permutation entropy of T is defined as

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Allow}(T, n)|,$$

provided that the limit exists.

Symbolic dynamical systems

- The idea is to discretize dynamical systems.
- The set X is partitioned into subsets $P_1, \ldots, P_k$.
- A point $x \in X$ is coded by a right-infinite word $(a_n)_{n \in \mathbb{N}}$:

$$\forall n \in \mathbb{N}, \; a_n = i \text{ whenever } T^n(x) \in P_i.$$

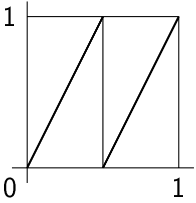
- If a point $x \in X$ is coded by $(a_n)_{n \in \mathbb{N}}$, then its image $T(x)$ is coded by $(a_{n+1})_{n \in \mathbb{N}}$.

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle code}\downarrow & & \downarrow{\scriptstyle code} \\ A^{\mathbb{N}} & \xrightarrow{\sigma} & A^{\mathbb{N}} \end{array}$$

- We are interested in determining which sequences can arise in this way.

Binary representation of numbers

Let $T\colon [0,1) \to [0,1),\ x \mapsto \{2x\}$.

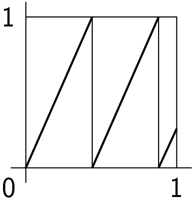


We partition $[0,1)$ into the 2 subintervals $[0,\frac{1}{2})$ and $[\frac{1}{2},1)$, which are coded by 0 and 1 respectively.

Then the coding of a real number x just corresponds to its binary expansion.

Representation of numbers in a real base β

Let $\beta > 1$ a real number and $T_\beta : [0,1) \to [0,1),\ x \mapsto \{\beta x\}$.



$\beta = \sqrt{5}$

We partition $[0,1)$ into the $\lceil \beta \rceil$ subintervals

$$\Big[0, \frac{1}{\beta}\Big),\ \Big[\frac{1}{\beta}, \frac{2}{\beta}\Big),\ \ldots,\ \Big[\frac{\lceil \beta \rceil - 1}{\beta}, 1\Big),$$

which are coded by $0, 1, \ldots, \lceil \beta \rceil - 1$ respectively.

In this case the coding of a real number x corresponds to its β-expansion.

Symbolic dynamical systems

- X is a subset of $A^{\mathbb{N}}$ stable under the shift operator σ:

$$\sigma((a_n)_{n\in\mathbb{N}}) = (a_{n+1})_{n\in\mathbb{N}}$$

- Denote by σ_X the restriction of the operator σ to X.
- If X is also compact then (X, σ_X) is called a symbolic dynamical system, a shift space, or simply a shift.
- A shift can also be described as a set $X_{\mathcal{F}}$ of all sequences avoiding the finite blocks in $\mathcal{F}$.
- $(A^{\mathbb{N}}, \sigma)$ is called the full shift.

Entropy in shifts (X, σ_X)

- $\text{Fact}_n(X)$ is the number of factors of length n that appear in some $x \in X$.
- Typically, $|\text{Fact}_n(X)|$ grows like 2^{cn} for some constant c.
- The entropy of σ_X is given by

$$\lim_{n\to\infty} \frac{1}{n} \log |\text{Fact}_n(X)|$$

- We can equip $A^{\mathbb{N}}$, and hence any shift space, with a total order, as the lexicographic order for example.
- Using Bandt-Keller-Pompe's result, an alternative way to compute the entropy is given by

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Allow}(\sigma_X, n)|$$

Uncountably many forbidden permutations

- The same result implies that not all permutations are realizable in such a dynamical system.
- In fact, in general, there are much more forbidden permutations than realizable permutations.
- Quote from Elizalde: *"Understanding the forbidden patterns of chaotic maps is important because the absence of these patterns is what distinguishes sequences generated by chaotic maps from random sequences."*

Part I: Permutations in full shifts

Forbidden patterns and shift systems [Amigó-Elizalde-Kennel 2008]
The number of permutations realized by a shift [Elizalde 2009]

The full shift over k symbols

- Let $A_k = \{0, 1, \dots, k-1\}$ and $\sigma_k : A_k^{\mathbb{N}} \to A_k^{\mathbb{N}}$ denote the shift operator.
- Elements in $A_k^{\mathbb{N}}$ are ordered by the lexicographic order:

$$a_1 a_2 \cdots <_{\text{lex}} b_1 b_2 \cdots \iff \exists i \geq 1,\ a_1 \cdots a_{i-1} = b_1 \cdots b_{i-1} \text{ and } a_i < b_i$$

- Study the permutations realizable in full shifts $(A_k^{\mathbb{N}}, \sigma_k)$, that is, the sets $\mathrm{Allow}(\sigma_k)$.
- In particular, for a given permutation π, compute the quantity

$$N_+(\pi) = \min\{k \geq 1 : \pi \in \mathrm{Allow}(\sigma_k)\}$$

which is the number of symbols needed in order to realize π.

Example ($\pi = 4217536 \in \mathcal{S}_7$)

Then $\text{Pat}(\sigma_3, 7, 210221220\cdots) = \pi$ since

$210221220\cdots$	4
$10221220\cdots$	2
$0221220\cdots$	1
$221220\cdots$	7
$21220\cdots$	5
$1220\cdots$	3
$220\cdots$	6

Another way to see it is:

$$2\,1\,0\,2\,2\,1\,2\,2\,0\cdots$$

$$4\,2\,1\,7\,5\,3\,6$$

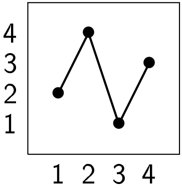
In fact, to realize the permutation π, one needs 3 symbols, so that $N_+(\pi) = 3$.

Computing $N_+(\pi)$

- Associated with $\pi \in \mathcal{S}_n$, we consider the circular permutation (or n-cycle)
$$\hat{\pi} = (\pi(1)\pi(2)\cdots\pi(n)),$$
that is, $\hat{\pi}(\pi(i)) = \pi(i+1)$ for $1 \leq i < n$, and $\hat{\pi}(\pi(n)) = \pi(1)$.
- Idea: Count the number of descents in $\hat{\pi}$.
- A descent in a permutation $\pi \in \mathcal{S}_n$ is an index $1 \leq i < n$ such that $\pi(i) > \pi(i+1)$.

Example

If we represent the permutation $\pi = 2413$, we see that it has one descent:



Computing $N_+(\pi)$

Theorem (Elizalde 2009)

For any $\pi \in \mathcal{S}_n$, the minimal number k of distinct symbols of a sequence w satisfying $\mathrm{Pat}(w, \sigma_k, n) = \pi$ is

$$N_+(\pi) = 1 + \mathrm{des}(\hat{\pi}) + \epsilon_+(\pi)$$

where $\mathrm{des}(\hat{\pi})$ is the number of descents in $\hat{\pi}$ with $\pi(1)$ removed, and

$$\epsilon_+(\pi) = \begin{cases} 1 & \text{if } \pi \text{ ends with } 21 \text{ or with } (n-1)n, \\ 0 & \text{otherwise.} \end{cases}$$

Computing $N_+(\pi)$: Sketch of the proof

- If $\begin{cases} a_i a_{i+1} \cdots <_{\text{lex}} a_j a_{j+1} \cdots \\ a_i = a_j \end{cases}$ then $a_{i+1} a_{i+2} \cdots <_{\text{lex}} a_{j+1} a_{j+2} \cdots$
- Now suppose that $a_1 a_2 \cdots \in A_k^{\mathbb{N}}$ realizes the permutation $\pi \in \mathcal{S}_n$, that is, $\text{Pat}(\sigma_k, n, a_1 a_2 \cdots) = \pi$.
- If $\begin{cases} \pi(i) < \pi(j) \\ a_i = a_j \\ i, j < n \end{cases}$ then $\pi(i+1) < \pi(j+1)$.
- We make use of $\hat{\pi}$ with the contrapositive statement.
- If $\begin{cases} \pi(i) + 1 = \pi(j) \\ \pi(i+1) = \hat{\pi}(\pi(i)) > \hat{\pi}(\pi(j)) = \pi(j+1) \\ i, j < n \end{cases}$ then $a_i < a_j$.
- So, for each descent in $\hat{\pi}$ with $\pi(1)$ removed, we need one more symbol.

Example ($\pi = 4217536 \in \mathcal{S}_7$)

One has $\hat{\pi} = (4217536) = 7162\underline{3}45$ and $\mathrm{des}(\hat{\pi}) = 2$.

	$\hat{\pi} =$	7	1	6	2	3	$\underline{4}$	5
Finding digits								
	$\pi =$	4	2	1	7	5	3	6
Placing digits								

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	$\hat{\pi} =$	7	1	6	2	3	$\underline{4}$	5
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Finding digits		0	1	1	2	2		2
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Placing digits			1	0			1	

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Computing $N_+(\pi)$

Theorem (Elizalde 2009)

For any $\pi \in \mathcal{S}_n$, the minimal number k of distinct symbols of a sequence w satisfying $\mathrm{Pat}(w, \sigma_k, n) = \pi$ is

$$N_+(\pi) = 1 + \mathrm{des}(\hat{\pi}) + \epsilon_+(\pi)$$

where $\mathrm{des}(\hat{\pi})$ is the number of descents in $\hat{\pi}$ with $\pi(1)$ removed and

$$\epsilon_+(\pi) = \begin{cases} 1 & \text{if } \pi \text{ ends with } 21 \text{ or with } (n-1)n, \\ 0 & \text{otherwise.} \end{cases}$$

Permutations in full shifts

- The shortest forbidden permutations of $A_k^{\mathbb{N}}$, have length $k+2$.
- For every $\pi \in \mathcal{S}_n$ we have $N_+(\pi) \leq n-1$.
- There are exactly 6 permutations π in $\mathcal{S}_n$ such that $N_+(\pi) = n-1$:

$$\begin{array}{ll} 1n2(n-1)3(n-2)\dots, & \dots(n-2)3(n-1)2n1, \\ n1(n-1)2(n-2)3\dots, & \dots3(n-2)2(n-1)1n, \\ \dots4(n-1)3n21, & \dots(n-3)2(n-2)1(n-1)n. \end{array}$$

Permutations in full shifts

- In fact, Elizalde shows much more by proving a closed formula for the number $a_{n,\ell}$ of permutations π of length n for which $N_+(\pi) = \ell$, for any n and ℓ.
- In particular, for each fixed ℓ, $a_{n,\ell} \sim n\ell^{n-1}$ as $n \to \infty$.
- Then, for each k,

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Allow}(\sigma_k, n)| = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\ell=1}^{k} a_{n,\ell} \right) = \log k,$$

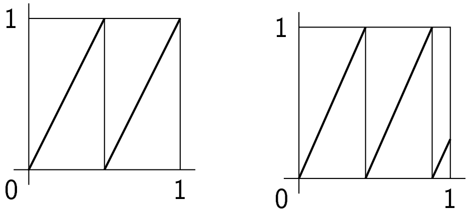
in accordance with Bandt-Keller-Pompe's theorem.

Part II: Permutations in β-shifts

Permutations and β-shifts [Elizalde 2011]

Permutations in β-shifts

- For $\beta > 1$ we study the dynamical systems $([0,1), T_\beta)$ where $T_\beta\colon [0,1) \to [0,1),\ x \mapsto \{\beta x\}$.



$\beta = 2$ $\qquad$ $\beta = \sqrt{5}$

- Study the realizable/forbidden permutations.

The β-shift

- Instead of numbers $x \in [0,1)$, we will rather consider their β-expansions, denoted by $d_\beta(x)$.
- We let Ω_β denote the topological closure of the set $\{d_\beta(x) \colon x \in [0,1)\}$ and $\sigma_\beta \colon \Omega_\beta \to \Omega_\beta,\ (a_m) \mapsto (a_{m+1})$.
- The map σ_β is continuous and Ω_β is a compact metric space, hence the β-shift $(\Omega_\beta, \sigma_\beta)$ is a topological dynamical system.
- The case $\beta \in \mathbb{N}$ corresponds to full shifts.

$\mathrm{Allow}(T_\beta) = \mathrm{Allow}(\sigma_\beta)$

- Key observation: $x < y \iff d_\beta(x) <_{\text{lex}} d_\beta(y)$.
- The following diagram commutes

$$\begin{array}{ccc} [0,1) & \xrightarrow{T_\beta} & [0,1) \\ \downarrow d_\beta & & \downarrow d_\beta \\ \Omega_\beta & \xrightarrow{\sigma_\beta} & \Omega_\beta \end{array}$$

- Thus, T_β and σ_β are order-isomorphic, and, for all $x \in [0,1)$ and all $n \geq 1$, we have

$$\mathrm{Pat}(T_\beta, n, x) = \mathrm{Pat}(\sigma_\beta, n, d_\beta(x)),$$

with the lexicographic order on Ω_β.

The shift complexity

- If $1 < \beta \leq \beta'$ then $\Omega_\beta \subseteq \Omega_{\beta'}$ and $\mathrm{Allow}(T_\beta) \subseteq \mathrm{Allow}(T_{\beta'})$.
- Compute $B_+(\pi) = \inf\{\beta > 1 \colon \pi \in \mathrm{Allow}(T_\beta)\}$.
- This quantity is called the (positive) shift complexity of π.

Example

For $n = 2$, one has $B_+(12) = B_+(21) = 1$.
For $n = 3$, one has

$$B_+(132) = B_+(213) = B_+(321) = \frac{1+\sqrt{5}}{2}$$

and

$$B_+(123) = B_+(231) = B_+(312) = 1.$$

Computing the shift complexity

- For $a = a_1 a_2 \cdots$ such that $a = \sup_{j \geq 1} a_j a_{j+1} \cdots \neq \overline{0}$, let $b_+(a)$ be the unique solution $\beta \geq 1$ of
$$\sum_{j=1}^{\infty} \frac{a_j}{\beta^j} = 1.$$
By convention, $b_+(\overline{0}) = 1$.
- If a is eventually periodic then $b_+(a)$ is the unique real root greater than or equal to 1 of a polynomial.

Computing the shift complexity

- For $\pi \in \mathcal{S}_n$, define $z_1 z_2 \cdots z_{n-1}$ as in the case of full shifts.
- Let $m = \pi^{-1}(n)$ and $\ell = \pi^{-1}(\pi(n)-1)$ if $\pi(n) \neq 1$.

Theorem (Elizalde 2011)

Let $\pi \in \mathcal{S}_n$ and $\beta > 1$. Then $\pi \in \mathrm{Allow}(T_\beta) \iff \beta > b_+(a)$ where

$$a = \begin{cases} z_{[m,n)}\overline{z_{[\ell,n)}} & \text{if } \pi(n) \neq 1, \\ z_{[m,n)}\overline{0} & \text{if } \pi(n) = 1 \text{ and } \pi(n-1) \neq 2, \\ z'_{[m,n)}\overline{0} & \text{if } \pi(n) = 1 \text{ and } \pi(n-1) = 2. \end{cases}$$

where for $1 \leq j < n$, $z'_j = z_j + 1$. In particular, $B_+(\pi) = b_+(a)$.

Theorem (Elizalde 2011)

We always have $\pi \notin \mathrm{Allow}(T_{B_+(\pi)})$. So $N_+(\pi) = 1 + \lfloor B_+(\pi) \rfloor$.

Minimal shift complexity

The only permutations $\pi \in \mathcal{S}_n$ satisfying $B_+(\pi) = 1$ are

$$(c+1)(c+2)\dots n12\dots c$$

for any fixed $1 \leq c \leq n$.

Example

We already saw that

$$B_+(12) = B_+(21) = 1$$

and

$$B_+(123) = B_+(231) = B_+(312) = 1.$$

Maximal shift complexity

For $n = 3$, there is 3 permutations of maximal complexity.

Theorem (Elizalde 2011)

For $\pi \in \mathcal{S}_n \setminus \{\rho_n\}$ with $n \geq 4$, we have $B_+(\pi) < B_+(\rho_n)$ where

$$\rho_n = \begin{cases} 1n2(n-1)\ \ldots\ \frac{n}{2}\,\frac{n+2}{2} & \text{if } n \text{ is even} \\ 1n2(n-1)\ \ldots\ \frac{n-1}{2}\,\frac{n+3}{2}\,\frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, $B_+(\rho_n) \in [n-2, n-1)$.

Example

We have $\rho_4 = 1423$ and $B_+(\rho_4) = \frac{3+\sqrt{5}}{2} = 2.61\ldots$

$B_+(\rho_n)$ is the threshold

Recall that $\pi \notin \mathrm{Allow}(T_{B_+(\pi)})$. Therefore we get

Corollary

For $n \geq 4$, we have $\mathcal{S}_n \subseteq \mathrm{Allow}(T_\beta) \iff \beta > B_+(\rho_n)$.

Example (continued)

For $\beta > 2.61\ldots$, the β-shift allows all permutations of length ≤ 4.

Corollary

For a fixed $\beta > 1$, the length of the shortest forbidden permutation of T_β is the integer $n \geq 2$ defined by $B_+(\rho_{n-1}) < \beta \leq B_+(\rho_n)$.

Part III: Permutations and negative β-shifts

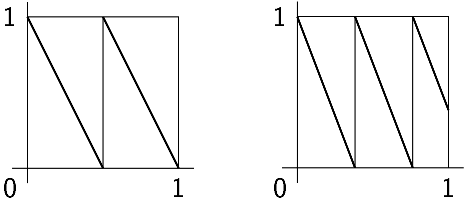
Patterns of negative shifts and beta-shifts [Elizalde-Moore]
Permutations and negative beta-shifts [Charlier-Steiner]

Negative β-shifts

- Let $\beta > 1$. We study the map

$$T_{-\beta} \colon (0,1] \to (0,1],\ x \mapsto \lfloor \beta x \rfloor + 1 - \beta x.$$

- Generalization of T_β as $T_{-\beta}(x) = \{-\beta x\}$ except for finitely many points.



$\beta = 2$ $\qquad$ $\beta = \frac{3+\sqrt{5}}{2}$

Negative β-shifts

- Again, instead of numbers $x \in (0, 1]$, we consider their $(-\beta)$-expansions, denoted by $d_{-\beta}(x)$.
- $\Omega_{-\beta}$ is the closure of $\{d_{-\beta}(x) : x \in (0, 1]\}$.
- The shift map is $\sigma_{-\beta} : \Omega_{-\beta} \to \Omega_{-\beta}, \ (a_m) \mapsto (a_{m+1})$.

Permutations in negative β-shifts

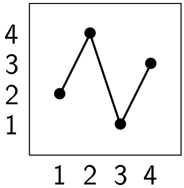
- Key observation: $x < y \iff d_{-\beta}(x) <_{\mathrm{alt}} d_{-\beta}(y)$.
- Here we use the alternating lexicographic order for sequences:

$$a_1 a_2 \cdots <_{\mathrm{alt}} b_1 b_2 \cdots \iff \exists i \geq 1,\ a_1 \cdots a_{i-1} = b_1 \cdots b_{i-1}$$
$$\text{and } \begin{cases} a_i < b_i & \text{if } i \text{ is odd,} \\ a_i > b_i & \text{if } i \text{ is even.} \end{cases}$$

  For example, $1320\cdots <_{\mathrm{alt}} 1210\cdots <_{\mathrm{alt}} 1220\cdots$
- We have $\mathrm{Allow}(T_{-\beta}) = \mathrm{Allow}(\sigma_{-\beta})$ with the alternating lexicographic order on the $(-\beta)$-shift.

Count the number of ascents

- We have to adapt the arguments from the full shift case.
- We consider again $\hat{\pi} = (\pi(1)\pi(2)\cdots\pi(n))$.
- Idea: For each ascent in $\hat{\pi}$ with $\pi(1)$ removed, we need one more symbol.



Theorem (Charlier-Steiner, Elizalde-Moore)

Let $\pi \in \mathcal{S}_n$. Then the minimal number of symbols of a sequence w satisfying $\mathrm{Pat}(\sigma_k, n, w) = \pi$ w.r.t. the alternating lexicographic order is

$$N_-(\pi) = 1 + \mathrm{asc}(\hat{\pi}) + \epsilon_-(\pi),$$

where $\mathrm{asc}(\hat{\pi})$ is the number of ascents in $\hat{\pi}$ with $\pi(1)$ removed and

$$\epsilon_-(\pi) = \begin{cases} 1 & \text{if some condition on } \pi \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular $N_-(\pi) \leq n-1$ for all $\pi \in \mathcal{S}_n$, $n \geq 3$.

For $n \geq 4$, there are exactly 4 permutations $\pi \in \mathcal{S}_n$ with $N_-(\pi) = n-1$:

$$12\ldots n, \quad 12\ldots(n-2)n(n-1), \quad n(n-1)\ldots 1, \quad n(n-1)\ldots 312.$$

Permutations in negative β-shifts

- Study the permutations realizable/forbidden in negative β-shifts.
- Compute the negative shift complexity

$$B_-(\pi) = \inf\{\beta > 1 \colon \pi \in \mathrm{Allow}(T_{-\beta})\}.$$

Computing the negative shift complexity

Theorem (Charlier-Steiner, Elizalde-Moore)

Let $\pi \in \mathcal{S}_n$ *and* $\beta > 1$. *Then* $\pi \in \mathrm{Allow}(T_{-\beta}) \iff \beta > b_-(a)$ *where*

$$a = \begin{cases} z_{[m,n)}\,\overline{z_{[\ell,n)}} & \text{if } n-m \text{ is even, } \pi(n) \neq 1, \text{ and } (\star), \\ \min_{0\leq i<|r-\ell|} z^{(i)}_{[m,n)}\,\overline{z^{(i)}_{[\ell,n)}} & \text{if } n-m \text{ is even, } \pi(n) \neq 1, \text{ and } \neg(\star), \\ \overline{z_{[m,n)}0} & \text{if } n-m \text{ is even and } \pi(n) = 1, \\ z_{[m,n)}\,\overline{z_{[r,n)}} & \text{if } n-m \text{ is odd and } \neg(\star), \\ \min_{0\leq i<|r-\ell|} z^{(i)}_{[m,n)}\,\overline{z^{(i)}_{[r,n)}} & \text{if } n-m \text{ is odd and } (\star). \end{cases}$$

In particular $B_-(\pi) = b_-(a)$.

Theorem (Charlier-Steiner, Elizalde-Moore)

We have $N_-(\pi) = 1 + \lfloor B_-(\pi) \rfloor$.

Minimal negative shift complexity

Theorem (Charlier-Steiner)

If $a >_{\mathrm{alt}} \varphi^{\omega}(0)$ where $\varphi\colon 0 \mapsto 1,\ 1 \mapsto 100$, then $B_{-}(\pi)$ is a Perron number, i.e., an algebraic integer $\beta > 1$ all of whose Galois conjugates α satisfy $|\alpha| < \beta$.

Moreover, $B_{-}(\pi) = 1 \iff a = \overline{\varphi^{k}(0)}$ for some $k \geq 0$.

Comparing the positive and negative β-shifts

$B_{\pm}(\pi)$	root of	π, negative beta-shift	π, positive beta-shift
1	$\beta - 1$	$12, 21$ $123, 132, 213, 231, 321$ $1324, 1342, 1432, 2134$ $2143, 2314, 2431, 3142$ $3214, 3241, 3421, 4213$	$12, 21$ $123, 231, 312$ $1234, 2341, 3412, 4123$
1.465	$\beta^3 - \beta^2 - 1$		$1342, 2413, 3124, 4231$
1.618	$\beta^2 - \beta - 1$	312 $1423, 3412, 4231$	$132, 213, 321$ $1243, 1324, 2431, 3142, 4312$
1.755	$\beta^3 - 2\beta^2 + \beta - 1$	$2341, 2413, 3124, 4123$	
1.802	$\beta^3 - 2\beta^2 - 2\beta + 1$		4213
1.839	$\beta^3 - \beta^2 - \beta - 1$	4132	$1432, 2143, 3214, 4321$
2	$\beta - 2$	$1234, 1243$	$2134, 3241$
2.247	$\beta^3 - 2\beta^2 - \beta + 1$	4321	4132
2.414	$\beta^2 - 2\beta - 1$		$2314, 3421$
2.618	$\beta^2 - 3\beta + 1$		1423
2.732	$\beta^2 - 2\beta - 2$	4312	

Some open problems

- Count all permutations with $B_-(\pi) \leq N$ or $B_-(\pi) < N$, in particular with $B_-(\pi) = 1$. From Bandt-Keller-Pompe's theorem we know that

$$\lim_{n\to\infty} \frac{1}{n} \log \#\{\pi \in \mathcal{S}_n : B_-(\pi) \leq \beta\} = \log \beta$$

  What are the precise asymptotics of

$$c_n = \#\{\pi \in \mathcal{S}_n : B_-(\pi) = 1\}?$$

  We have $(c_n)_{n\geq 2} = 2, 5, 12, 19, 34, 57, 82, 115, \dots$

- Describe the permutations given by the transformations

$$T_{\beta,\alpha} : [0,1) \to [0,1), \; x \mapsto \beta x + \alpha - \lfloor \beta x + \alpha \rfloor.$$

References

- A. Rényi, Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar* **8**, 477–493, 1957.
- Ch. Bandt, G. Keller and B. Pompe, Entropy of interval maps via permutations. *Nonlinearity* **15(5)**, 1595–1602, 2002.
- J.M. Amigó, S. Elizalde, G. Kennel, Forbidden patterns and shift systems. *J. Combin. Theory Ser. A* **115(3)**, 485–504, 2008.
- S. Elizalde, The number of permutations realized by a shift. *SIAM J. Discrete Math.* **23(2)**, 765–786, 2009.
- S. Ito and T. Sadahiro, Beta-expansions with negative bases. *Integers* **9(A22)**, 239–259, (2009).
- S. Elizalde, Permutations and β-shifts. *J. Combin. Theory Ser. A* **118(8)**, 2474–2497, 2011.
- S. Elizalde and K. Moore, Patterns of negative shifts and beta-shifts, working paper.
- É. Charlier and W. Steiner, Permutations and negative beta-shifts, working paper.