Generating expansions in Cantor real bases via a transducer

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Cantor real bases and alternate bases

A Cantor real base is a sequence $\boldsymbol{B} = (\beta_n)_{n\geq 0}$ of real numbers such that

- $\beta_n > 1$ for all n
- $\prod_{n=0}^{\infty} \beta_n = \infty$.

A $\boldsymbol{B}$-representation of a real number x is an infinite sequence $a = (a_n)_{n\geq 0}$ of integers such that

$$x = \sum_{n=0}^{\infty} \frac{a_n}{\beta_0 \cdots \beta_n}.$$

An alternate base is a periodic Cantor base. In this case, we simply write $\boldsymbol{B} = (\beta_0, \ldots, \beta_{p-1})$ and we use the convention that $\beta_n = \beta_{n \bmod p}$ for all $n \geq 0$.

Cantor 1869, Rényi 1959, Galambos 1976

Greedy and quasi-greedy

For $x \in [0,1]$, a distinguished $\boldsymbol{B}$-representation

$$d_{\boldsymbol{B}}(x) = (a_n)_{n \geq 0},$$

called the greedy $\boldsymbol{B}$-expansion of x, is obtained from the greedy algorithm:

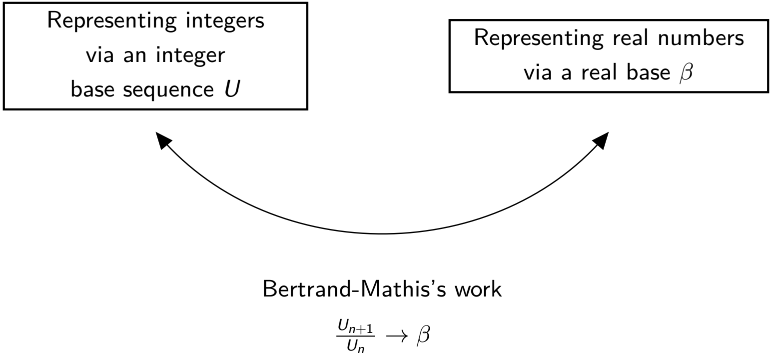
- We first set $r_0 = x$.
- Then set $a_n = \lfloor \beta_n r_n \rfloor$ and $r_{n+1} = \beta_n r_n - a_n$ for $n \geq 0$.

Another distinguished $\boldsymbol{B}$-representation is the quasi-greedy $\boldsymbol{B}$-expansion. It is obtained recursively:

$$d^*_{\boldsymbol{B}}(x) = \begin{cases} d_{\boldsymbol{B}}(x), & \text{if } d_{\boldsymbol{B}}(x) \text{ does not end in } 0^\omega; \\ a_0 a_1 \cdots a_{\ell-2}(a_{\ell-1}-1) d^*_{\sigma^\ell(\boldsymbol{B})}(1), & \text{if } d_{\boldsymbol{B}}(x) = a_0 a_1 \cdots a_{\ell-1} 0^\omega,\ a_{\ell-1} \neq 0, \end{cases}$$

where σ is the shift operator on sequences, so that $\sigma^\ell(\boldsymbol{B}) = (\beta_n)_{n \geq \ell}$.

Motivation



Representing integers via an integer base sequence U

Representing real numbers via a real base β

Bertrand-Mathis's work

$\frac{U_{n+1}}{U_n} \to \beta$

In the general case, there is a similar relationship with representations of real numbers via some alternate base $\boldsymbol{B} = (\beta_0, \ldots, \beta_{p-1})$.

How alternate real bases occurred to me

Consider $U = (U_n)_{n\geq 0}$ starting with $1, 2, 3, 7$ and such that $U_n = 3U_{n-2} + U_{n-4}$ for $n \geq 4$.

We get $U = (1, 2, 3, 7, 10, 23, 33, 76, 109 \ldots)$.

The quotient $\frac{U_{n+1}}{U_n}$ does not have a limit, but alternates between to values

$$\alpha = \lim_{n\to+\infty} \frac{U_{2n+1}}{U_{2n}} = \frac{1+\sqrt{13}}{2} \quad \text{and} \quad \beta = \lim_{n\to+\infty} \frac{U_{2n+2}}{U_{2n+1}} = \frac{5+\sqrt{13}}{6}.$$

Now, if we take a look at the largest representations of each length, we get

$U_n - 1$	0	1	2	6	9	22	32	75	108	$\cdots$
$\mathrm{rep}_U(U_n - 1)$	ε	1	10	200	1010	20010	101010	2001010	10101010	$\cdots$

These maximal words are the prefixes of the two ultimately periodic infinite words $(10)^\omega$ and $200(10)^\omega$.

These infinite words are not random. To understand them, we have to represent 1 using the two real bases α, β alternatively.

Let $\alpha = \frac{1+\sqrt{13}}{2} \sim 2.30$ and $\beta = \frac{5+\sqrt{13}}{6} \sim 1.43$.

Consider the alternate base $\boldsymbol{B} = (\alpha, \beta) = (\alpha, \beta, \alpha, \beta, \ldots)$.

$r_0 = 1$	$\alpha r_0 = \frac{1+\sqrt{13}}{2}$	$\varepsilon_0 = \lfloor \alpha r_0 \rfloor = 2$
$r_1 = \alpha r_0 - \varepsilon_0 = \frac{-3+\sqrt{13}}{2}$	$\beta r_1 = \frac{-1+\sqrt{13}}{6}$	$\varepsilon_1 = \lfloor \beta r_1 \rfloor = 0$
$r_2 = \beta r_1 - \varepsilon_1 = \frac{-1+\sqrt{13}}{6}$	$\alpha r_2 = 1$	$\varepsilon_2 = \lfloor \alpha r_2 \rfloor = 1$
$r_3 = \alpha r_2 - \varepsilon_2 = 0$	$\beta r_3 = 0$	$\varepsilon_3 = \lfloor \beta r_3 \rfloor = 0$

Then $d_{\boldsymbol{B}}(1) = 2010^{\omega}$.

In particular, we have $1 = \frac{2}{\alpha} + \frac{1}{\alpha^2 \beta}$.

Now consider the shifted alternate base $\sigma(\boldsymbol{B}) = (\beta, \alpha) = (\beta, \alpha, \beta, \alpha, \ldots)$.

$r_0 = 1$	$\beta r_0 = \frac{5+\sqrt{13}}{6}$	$\varepsilon_0 = \lfloor \beta r_0 \rfloor = 1$
$r_1 = \beta r_0 - \varepsilon_0 = \frac{-1+\sqrt{13}}{6}$	$\alpha r_1 = 1$	$\varepsilon_1 = \lfloor \alpha r_1 \rfloor = 1$
$r_2 = \alpha r_1 - \varepsilon_1 = 0$	$\beta r_2 = 0$	$\varepsilon_2 = \lfloor \beta r_2 \rfloor = \lfloor 0 \rfloor = 0$

Then $d_{\sigma(\boldsymbol{B})}(1) = 110^{\omega}$.

In particular, we have $1 = \frac{1}{\beta} + \frac{1}{\beta\alpha}$.

We find the two words followed by the maximal words in the numeration language that we have seen before by using the quasi-greedy expansions of 1:

$$d^*_{S(B)}(1) = 10d^*_{S(B)}(1) = 1010d^*_{d_{S(B)}}(1) = \cdots = (10)^{\omega}$$

and

$$d^*_{\sigma(\boldsymbol{B})}(1) = 200d^*_{S(\boldsymbol{B})}(1) = 200(10)^{\omega}.$$

Some references

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Starting point of this work

Let now $\boldsymbol{B} = (\beta_n)_{n\geq 0} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ be the Thue-Morse sequence over $\{\alpha, \beta\}$:

$$\beta_n = \begin{cases} \alpha & \text{if the binary expansion of } n \text{ has an even number of 1's,} \\ \beta & \text{otherwise.} \end{cases}$$

We compute $d_{\boldsymbol{B}}(1) = 20010110^{\omega}$.

$r_0 = 1$	$\alpha r_0 = \frac{1+\sqrt{13}}{2}$	$\varepsilon_0 = \lfloor \alpha r_0 \rfloor = 2$
$r_1 = \alpha r_0 - \varepsilon_0 = \frac{-3+\sqrt{13}}{2}$	$\beta r_1 = \frac{-1+\sqrt{13}}{6}$	$\varepsilon_1 = \lfloor \beta r_1 \rfloor = 0$
$r_2 = \beta r_1 - \varepsilon_1 = \frac{-1+\sqrt{13}}{6}$	$\beta r_2 = \frac{2+\sqrt{13}}{9}$	$\varepsilon_2 = \lfloor \beta r_2 \rfloor = 0$
$r_3 = \beta r_2 - \varepsilon_2 = \frac{2+\sqrt{13}}{9}$	$\alpha r_3 = \frac{5+\sqrt{13}}{6}$	$\varepsilon_3 = \lfloor \alpha r_3 \rfloor = \lfloor \frac{5+\sqrt{13}}{6} \rfloor = 1$
$r_4 = \alpha r_3 - \varepsilon_3 = \frac{-1+\sqrt{13}}{6}$	$\beta r_4 = \frac{2+\sqrt{13}}{9}$	$\varepsilon_4 = \lfloor \beta r_4 \rfloor = \lfloor \frac{2+\sqrt{13}}{9} \rfloor = 0$
$r_5 = \beta r_4 - \varepsilon_4 = \frac{2+\sqrt{13}}{9}$	$\alpha r_5 = \frac{5+\sqrt{13}}{6}$	$\varepsilon_5 = \lfloor \alpha r_5 \rfloor = \lfloor \frac{5+\sqrt{13}}{6} \rfloor = 1$
$r_6 = \alpha r_5 - \varepsilon_5 = \frac{-1+\sqrt{13}}{6}$	$\alpha r_6 = 1$	$\varepsilon_6 = \lfloor \alpha r_6 \rfloor = 1$
$r_7 = \alpha r_6 - \varepsilon_6 = 0$	$\beta r_7 = 0$	$\varepsilon_7 = \lfloor \beta r_7 \rfloor = 0$

Some questions we had in mind

- Do structural properties of the sequence $\boldsymbol{B}$ yield some structural properties of the $\boldsymbol{B}$-expansions?
- What is the dependence to the involved bases $\beta_0, \beta_1, \beta_2 \ldots$?
- Can we decide some properties of $\boldsymbol{B}$-expansions, when $\boldsymbol{B}$ enjoys some suitable combinatorial structure?
- Can we decide admissibility with respect to some aperiodic sequence $\boldsymbol{B}$?

First approach: Thue-Morse Cantor (integer) bases

Consider the base $\mathbf{T} = (2, 3, 3, 2, 3, 2, 2, 3, \ldots)$, the Thue-Morse sequence over $\{2, 3\}$.

Theorem (C-Popoli-Rigo)

For all $r \in [0, 1)$, the $\mathbf{T}$-expansion of r is the sequence

$$h_2(c_0)h_3(c_1)h_3(c_2)h_2(c_3)\cdots$$

obtained from $d_6(r) = c_0c_1c_2c_3\cdots$ by application of two specific substitutions h_2 and h_3 in the ordering given by the Thue-Morse sequence $\mathbf{T}$.

Definition of h_2

Each $c \in \{0, \ldots, 5\}$ can be uniquely decomposed as

$$c = a \cdot 3 + b \quad \text{with} \quad a \in \{0, 1\},\ b \in \{0, 1, 2\}.$$

The substitution h_2 is given by $h_2 : c \mapsto ab$, that is

$$h_2 \colon 0 \mapsto 00,\ 1 \mapsto 01,\ 2 \mapsto 02,$$
$$3 \mapsto 10,\ 4 \mapsto 11,\ 5 \mapsto 12.$$

Definition of h_3

Each $c \in \{0, \ldots, 5\}$ can be uniquely decomposed as

$$c = a \cdot 2 + b \quad \text{with} \quad a \in \{0, 1, 2\},\ b \in \{0, 1\}.$$

The substitution h_3 is given by $h_3 : c \mapsto ab$, that is

$$h_3 \colon 0 \mapsto 00,\ 1 \mapsto 01,\ 2 \mapsto 10,$$
$$3 \mapsto 11,\ 4 \mapsto 20,\ 5 \mapsto 21.$$

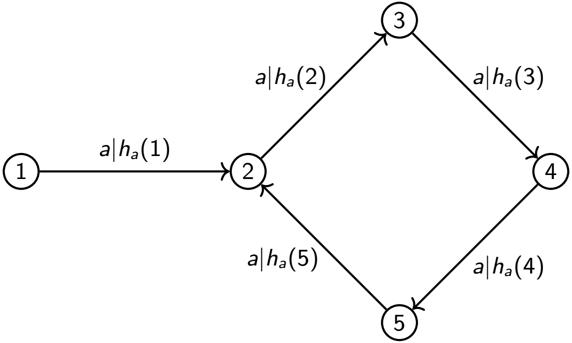
$$h_2\colon 0 \mapsto 00,\ 1 \mapsto 01,\ 2 \mapsto 02,\qquad h_3\colon 0 \mapsto 00,\ 1 \mapsto 01,\ 2 \mapsto 10,$$
$$3 \mapsto 10,\ 4 \mapsto 11,\ 5 \mapsto 12.\qquad 3 \mapsto 11,\ 4 \mapsto 20,\ 5 \mapsto 21.$$

For example, for $r = 932/3885$, we have

$$d_6(r) = 1(2345)^{\omega},$$

and then

$$\begin{aligned} d_{\mathsf{T}}(r) &= h_2(1)h_3(2)h_3(3)h_2(4)h_3(5)h_2(2)h_2(3)h_3(4)h_3(5)\cdots \\ &= 01101111210210202 1\cdots \end{aligned}$$



A finite transducer for $d_{\mathbf{T}}(932/3885)$, $a \in \{2, 3\}$.

We can modify it in order to make it letter-to-letter by replacing each label $a|h_a(j)$ with $\tau(a)|h_a(j)$, where τ is the TM substitution

$$\tau\colon 2 \mapsto 23, \quad 3 \mapsto 32.$$

Since TM substitution is 2-uniform, we can split these new edges into two edges in the natural way.

This yields a letter-to-letter transducer with 15 states with entries in $\{23, 32\}^{\omega}$.

This observation generalizes to finite sets E of integer bases other than $\{2, 3\}$ and other sequences of bases $\boldsymbol{B}$ over E.

Definition

Let $\delta \geq 2$ be an integer and let $A, E \subset \mathbb{N}_{\geq 2}$ be finite alphabets.
Let $\psi \colon A^* \to E^*$ be a substitution with the property that for each letter $a \in A$,

$$\text{if } \psi(a) = b_0 \cdots b_{\ell-1} \quad \text{then } \prod_{j=0}^{\ell-1} b_j = \delta.$$

For each letter $a \in A$, we define a substitution $h_a \colon \{0, \ldots, \delta - 1\}^* \to \{0, \ldots, (\max E) - 1\}^*$.
Every $c \in \{0, \ldots, \delta - 1\}$ can be uniquely written as

$$c = \sum_{j=0}^{\ell-1} c_j \cdot b_{j+1} \cdots b_{\ell-1}$$

where $c_0, \ldots, c_{\ell-1}$ are non-negative integers such that $c_j \in \{0, \ldots, b_j - 1\}$.
We then define $h_a(c) = c_0 \cdots c_{\ell-1}$.

As an example, take $\delta = 72$, $A = \{2, 3, 4\}$ and $E = \{2, 3, 4, 6\}$.

Consider the substitution $\psi \colon 2 \mapsto 634,\ 3 \mapsto 3243,\ 4 \mapsto 4332$.

For each $a \in A$, we have $h_a \colon \{0, \ldots, 71\}^* \to \{0, \ldots, 5\}^*$. For example, since

$$61 = 5 \cdot 12 + 0 \cdot 4 + 1,$$
$$61 = 2 \cdot 24 + 1 \cdot 12 + 0 \cdot 3 + 1,$$
$$61 = 3 \cdot 18 + 1 \cdot 6 + 0 \cdot 2 + 1,$$

we have $h_2(61) = 501$, $h_3(61) = 2101$ and $h_4(61) = 3101$.

Generalization to Cantor bases obtained as images under some well-structured morphisms

Consider a Cantor base $\boldsymbol{B} = \psi(a_0 a_1 a_2 \cdots)$ where $a_0 a_1 a_2 \cdots$ is any sequence over A.
For any real number $r \in [0, 1]$, if

$$d_\delta(r) = c_0 c_1 c_2 \cdots,$$

then

$$d_{\boldsymbol{B}}(r) = h_{a_0}(c_0) h_{a_1}(c_1) h_{a_2}(c_2) \cdots.$$

One transducer for uncountably many Cantor real bases

We make use of the notation $\mathrm{T}_\beta \colon [0,1] \to [0,1),\ r \mapsto \beta r - \lfloor \beta r \rfloor$.

Definition (Greedy transducer)

Let $E \subset \mathbb{R}_{>1}$ be a finite alphabet. We define an infinite transducer denoted by $\mathcal{T}_E$.

- The set of states is $[0,1]$;
- The input alphabet is E;
- The output alphabet is $\mathbb{N}$;
- For all states $r \in [0,1]$ and all letters $\beta \in E$, there is a transition

$$r \xrightarrow{\beta \mid \lfloor \beta r \rfloor} \mathrm{T}_\beta(r).$$

Proposition

When reading $\boldsymbol{B} \in E^{\mathbb{N}}$ *from a state* $r \in [0,1]$, *the transducer* $\mathcal{T}_E$ *outputs the greedy* $\boldsymbol{B}$*-representation of* r.

We make use of the notation $\mathsf{T}_\beta^*\colon [0,1] \to (0,1], \begin{cases} r \mapsto \beta r - \lceil \beta r - 1 \rceil, & \text{if } r \neq 0; \\ 0, & \text{if } r = 0. \end{cases}$

Definition (Quasi-greedy transducer)

Analogously, we define another infinite transducer denoted by $\mathcal{T}_E^*$, by the same setting but with transiions

$$r \xrightarrow{\beta \,|\, \lceil \beta\, r - 1 \rceil} \mathsf{T}_\beta^*(r).$$

Proposition

When reading $\boldsymbol{B} \in E^{\mathbb{N}}$ *from a state* $r \in [0,1]$, *the transducer* $\mathcal{T}_E^*$ *outputs the quasi-greedy* $\boldsymbol{B}$*-expansion of* r.

Definition

Considering a given $r \in [0,1]$ as initial state, we let $\mathcal{T}_{E,r}$ denote the part of $\mathcal{T}_E$ that can be reached from r. We define $\mathcal{T}_{E,r}^*$ analogously.

Proposition

The transducers $\mathcal{T}_{E,r}$ *and* $\mathcal{T}_{E,r}^*$ *are either both finite or both infinite.*

Conditions ensuring finiteness, and first applications

A Pisot number is a real algebraic integer which is greater than 1 and whose Galois conjugates all lies in the open unit disc.

Theorem (C-Popoli-Rigo)

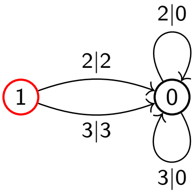
Let δ be an algebraic integer of degree d and let $E \subset \mathbb{Z}[\delta]$ be a finite set of Pisot numbers of degree d. For all $r \in \mathbb{Q}(\delta) \cap [0,1]$, the transducers $\mathcal{T}_{E,r}$ and $\mathcal{T}^*_{E,r}$ are finite.

Corollary

- *If $\boldsymbol{B} \in E^{\mathbb{N}}$ is an ultimately periodic sequence, then for all $r \in \mathbb{Q}(\delta) \cap [0,1]$, the sequences $d_{\boldsymbol{B}}(r)$ and $d^*_{\boldsymbol{B}}(r)$ are also ultimately periodic.*
- *If $\boldsymbol{B} \in E^{\mathbb{N}}$ is a b-automatic sequence for some integer base $b \geq 2$, then for all $r \in \mathbb{Q}(\delta) \cap [0,1]$, the sequences $d_{\boldsymbol{B}}(r)$ and $d^*_{\boldsymbol{B}}(r)$ are also b-automatic.*
- *If $\boldsymbol{B} \in E^{\mathbb{N}}$ is a morphic sequence, then for all $r \in \mathbb{Q}(\delta) \cap [0,1]$, the sequences $d_{\boldsymbol{B}}(r)$ and $d^*_{\boldsymbol{B}}(r)$ are also morphic.*

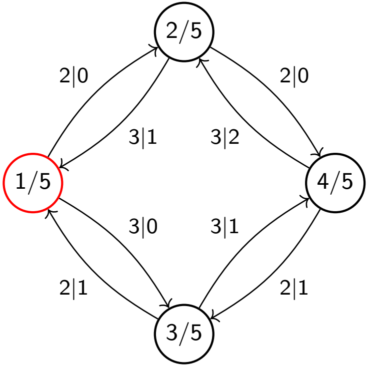
Examples

- For any finite set E of integer bases and any $r \in \mathbb{Q}$, the transducers $\mathcal{T}^*_{E,r}$ and $\mathcal{T}^*_{E,r}$ are finite.
- For $E = \{2, 3\}$, we get



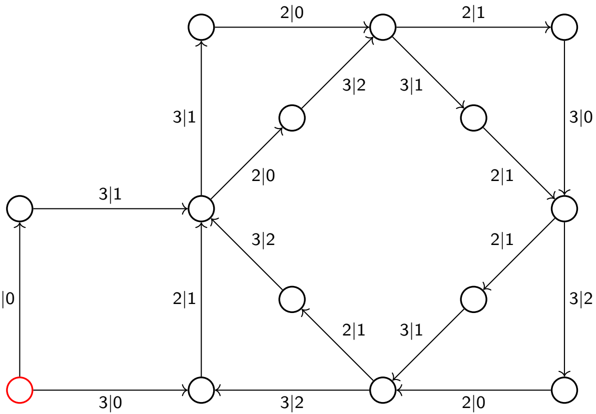
$\mathcal{T}_{E,1}$

$\mathcal{T}^*_{E,1}$



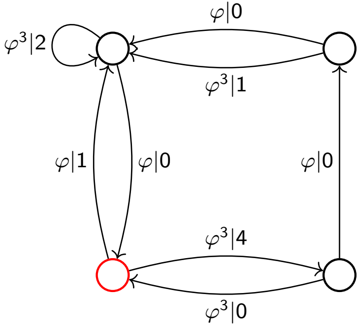
$\mathcal{T}_{E,\frac{1}{5}} = \mathcal{T}^*_{E,\frac{1}{5}}$

- For $r = \frac{932}{3885}$, we get a transducer $\mathcal{T}_{E,r}$ with 180 states.
- If we only consider inputs in $\{23, 32\}^*$, we obtain a transducer with only 14 states.

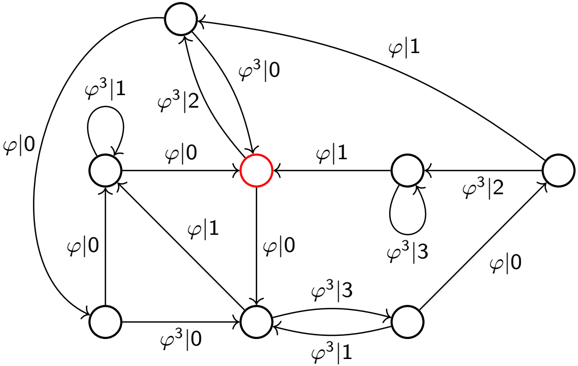


- Compared to the transucer obtained before with 15 states, two states have been merged.

- Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio and let $E = \{\varphi, \varphi^3\}$.



$\mathcal{T}^*_{E,1}$



$\mathcal{T}^*_{E,\frac{1}{2}}$

An infinite transducer $\mathcal{T}_{E,1}$

Let φ be the golden ratio and let $E = \{\varphi, 2\}$.

Here, φ has algebraic degree 2 whereas 2 has algebraic degree 1.

The set of states of the transducer that can be reached from 1 is infinite.

We can see this by reading the Cantor real base $\boldsymbol{B} = \varphi 2^{\omega}$ as input.

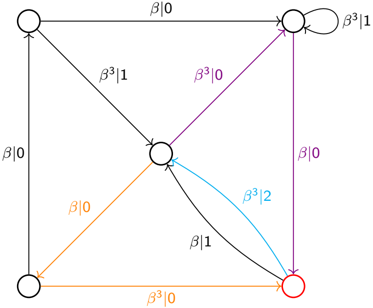
- After the first step using base φ, we reach the state $1/\varphi$.
- Thus, one has to represent $1/\varphi$ using only base 2.
- Since this number is irrational, its 2-expansion is aperiodic and we never encounter the same remainder twice in the greedy algorithm.

Cantor real bases giving rise to a periodic expansion of a given real number

Two-walk property

We say that $\mathcal{T}^*_{E,r}$ has the two-walk property if one can find two closed walks starting from the same state, with distinct inputs and the same output.

Let β be the smallest Pisot number, i.e., the real root of $X^3 - X - 1$. Consider the set $E = \{\beta, \beta^3\}$.

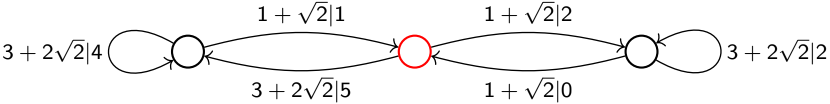


Let us write $u = \beta^3\beta\beta^3$ and $v = \beta^3\beta^3\beta$. Any Cantor real base $\boldsymbol{B} \in \{u, v\}^\omega$ leads to $d^*_{\boldsymbol{B}}(1) = (200)^\omega$.

Theorem (C-Popoli-Rigo)

The Cantor real bases $\boldsymbol{B} \in E^{\mathbb{N}}$ *such that* $d^*_{\boldsymbol{B}}(r)$ *is ultimately periodic are exactly the ultimately periodic ones if and only if the transducer* $\mathcal{T}^*_{E,r}$ *does not have the two-walk property.*

Consider the set $E = \{\beta, \beta^2\}$, where β is the Pisot number $1 + \sqrt{2}$ and its square β^2 is $3 + 2\sqrt{2}$.
The transducer $\mathcal{T}^*_{E,1}$ does not satisfy the two-walk property.



Proposition

The two-walk property is decidable in $\mathcal{O}((\#Q)^3)$ *steps, where* Q *is the set of states.*

Deciding admissibility in base $\boldsymbol{B}$

Main question: Given an infinite sequence $\mathbf{a}$, can we decide whether it is $\boldsymbol{B}$-admissible, that is, whether there exists some $x \in [0,1)$ such that $\mathbf{a} = d_{\boldsymbol{B}}(x)$?

Theorem (Parry 1960)

Let $\beta > 1$ be a real base. A sequence $\mathbf{a}$ over $\mathbb{N}$ is β-admissible if and only if $\sigma^n(\mathbf{a}) <_{\text{lex}} d^*_\beta(1)$ for all $n \in \mathbb{N}$.

Theorem (Caalim-Demegillo 2020, C-Cisternino 2021)

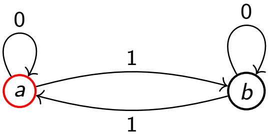
Let $\boldsymbol{B}$ be a Cantor real base. A sequence $\mathbf{a}$ over $\mathbb{N}$ is $\boldsymbol{B}$-admissible if and only if $\sigma^n(\mathbf{a}) <_{\text{lex}} d^*_{\sigma^n(\boldsymbol{B})}(1)$ for all $n \in \mathbb{N}$.

- In the Cantor real base case, these lexicographic conditions are far from being trivial to test, even in a simple situation where the Cantor base is the Thue–Morse word $\mathbf{T} = (2,3,3,2,3,2,2,3,\ldots)$ over $\{2,3\}$.
- This is due to the fact that the set $\{\sigma^n(\mathbf{T}) : n \geq 0\}$ is infinite since $\mathbf{T}$ is aperiodic.
- If $\mathbf{a}$ has no specific structure, then there is little chance that we can decide anything.

Automatic sequences

Let $b \geq 2$ be an integer. A sequence $(x_n)_{n\geq 0}$ is b-automatic if there exists a DFAO that produces x_n as ouput when the corresponding input is the b-ary expansion of n.

- For example, the TM sequence is 2-automatic.



n	0	1	2	3	4	5	6	7	8	$\cdots$
$\mathrm{rep}_2(n)$	ε	1	10	11	100	101	110	111	1000	$\cdots$
x_n	a	b	b	a	b	a	a	b	b	$\cdots$

Automatic sequences and logic

Theorem (Büchi 1960, Bruyère 1985)

A sequence is b-automatic if and only if it is definable by a first-order formula of $\langle \mathbb{N}, +, V_b \rangle$*, where* $V_b(0) = 1$ *and for* $n \neq 0$*,* $V_b(n) = i$ *if* b^i *is the highest power of* b *dividing* x*.*

Corollary

The first-order theory of the structure $\langle \mathbb{N}, +, V_b \rangle$ *is decidable.*

- This means that we can decide many combinatorial properties of automatic sequences.
- In our case, two sequences are involved: the base $\boldsymbol{B}$ and the given sequence $\mathbf{a}$ of which we want to decide the admissibility.
- The lexicographic conditions to test involve the shifts of both these sequences.

A technical decidability result

Theorem (C-Popoli-Rigo)

Let $E \subset \mathbb{R}_{>1}$ be a finite alphabet of real bases, let $\boldsymbol{B} \in E^{\mathbb{N}}$, and let $b \geq 2$ be an integer base.

Assume that, for each digit a that can occur in a $\boldsymbol{B}$-expansion, we are given a first-order formula $\varphi_a(j, n)$ of $\langle \mathbb{N}, +, V_b \rangle$ such that

$$\varphi_a(j, n) \text{ is true} \iff \text{the digit } a \text{ occurs at position } j \text{ in } d^*_{\sigma^n(\boldsymbol{B})}(1).$$

Then it is decidable whether any given b-automatic sequence over $\mathbb{N}$ is $\boldsymbol{B}$-admissible.

The alternate base case

Corollary

Let $b \geq 2$ be an integer, and let $\boldsymbol{B}$ be an alternate base such that $d^*_{\sigma^i(\boldsymbol{B})}(1)$ is b-automatic for all i.
Then it is decidable whether any given b-automatic sequence over $\mathbb{N}$ is $\boldsymbol{B}$-admissible.

A Parry alternate base is an alternate base $\boldsymbol{B}$ such that $d^*_{\sigma^i(\boldsymbol{B})}(1)$ is ultimately periodic for all i.

Corollary

Let $\boldsymbol{B}$ be a Parry alternate base and let $b \geq 2$ be an integer.
Then it is decidable whether any given b-automatic sequence over $\mathbb{N}$ is $\boldsymbol{B}$-admissible.

Since the finiteness of the transducer $\mathcal{T}^*_{E,1}$ implies that any alternate base $\boldsymbol{B}$ over E is Parry, we also get:

Corollary

Suppose that $\mathcal{T}^*_{E,1}$ is finite, let $\boldsymbol{B}$ be an alternate base over E, and let $b \geq 2$ be an integer.
Then it is decidable whether any given b-automatic sequence over $\mathbb{N}$ is $\boldsymbol{B}$-admissible.

A non-trivial example of decidability with an aperiodic Cantor real base

Proposition

Let $E = \{\beta, \beta^3\}$ *and let* $\boldsymbol{B} = (\beta^3, \beta, \beta^3, \beta^3, \beta^3, \beta, \beta^3, \beta^3, \beta, \beta^3, \beta, \beta^3, \beta^3, \beta^3, \beta, \beta^3, \beta, \beta^3, \ldots)$ *be the Cantor real base obtained as the Thue-Morse sequence over the blocks* $(\beta^3, \beta, \beta^3)$ *and* $(\beta^3, \beta^3, \beta)$*.*
It is decidable whether any given 2*-automatic sequence over* $\mathbb{N}$ *is* $\boldsymbol{B}$*-admissible.*

We used combined tools for proving this result:

- We took advantage of the 2-automaticity of $\boldsymbol{B}$ combined with structural properties of the transducer $\mathcal{T}^*_{E,1}$.
- The two-walk property was used, but it not sufficient. The choice of the blocks $(\beta^3, \beta, \beta^3)$ and $(\beta^3, \beta^3, \beta)$ is also important!
- This result generalizes to a family of sets E such that the transducer is finite and enjoys some specific (testable) feature.

Thank you!