A full characterization of Bertrand numeration systems

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Representing integers
via an integer
base sequence U

Representing real numbers
via a real base β

Bertrand-Mathis's work

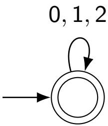
Representing integers in base 3

Any $n \in \mathbb{N}$ can be decomposed in a unique way as

$$n = \sum_{i=1}^{\ell} a_i 3^{\ell - i}$$

where $a_i \in \{0, 1, 2\}$ and $a_1 \neq 0$. We write $\mathrm{rep}_3(n) = a_1 \cdots a_\ell$.

The numeration language $\mathcal{N}_3$ is the set $0^*\mathrm{rep}_3(\mathbb{N})$, which is simply $\{0, 1, 2\}^*$.



Representing real numbers in base 3

Any $x \in [0,1)$ can be decomposed in a unique way as

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

where $a_i \in \{0,1,2\}$ and $a_i a_{i+1} a_{i+2} \cdots \neq 2^\omega$ for all i. We write $d_3(x) = a_1 a_2 a_3 \cdots$.

Define $D_3 = \{d_3(x) : x \in [0,1)\}$.

The topological closure of D_3 is called the 3-shift:

$$S_3 = \{w \in \{0,1,2\}^\omega : \mathrm{Fac}(w) \subseteq \mathrm{Fac}(D_3)\} = \{0,1,2\}^\omega.$$

Straightforward but crucial observation: $\mathrm{Fac}(S_3) = \mathcal{N}_3$.

Representing integers thanks to the Fibonacci sequence

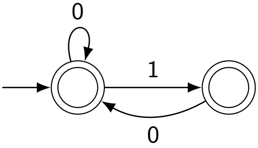
We let $F_0 = 1$, $F_1 = 2$ and $F_{i+2} = F_{i+1} + F_i$ for $i \geq 0$.

Any $n \in \mathbb{N}$ can be decomposed in a unique way as

$$n = \sum_{i=1}^{\ell} a_i F_{\ell - i}$$

where $a_i \in \{0, 1\}$ and $a_1 \neq 0$ with the condition that $a_i a_{i+1} \neq 11$. We write $\mathrm{rep}_F(n) = a_1 \cdots a_\ell$.

The numeration language $\mathcal{N}_F$ is the set $0^* \mathrm{rep}_F(\mathbb{N})$.



This numeration system is called the Zeckendorf numeration system.

Representing real numbers in base φ

Let $\varphi = \frac{1+\sqrt{5}}{2}$ (the golden mean).

Any $x \in [0, 1)$ can be decomposed in a unique way as

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\varphi^i}$$

where $a_i \in \{0, 1\}$, $a_i a_{i+1} \neq 11$ and $a_i a_{i+1} a_{i+2} \cdots \neq (10)^\omega$ for all i. We write $d_\varphi(x) = a_1 a_2 a_3 \cdots$.

Define $D_\varphi = \{d_\varphi(x) : x \in [0, 1)\}$.

The topological closure of D_φ is called the φ-shift:

$$S_\varphi = \{w \in \{0,1\}^\omega : \mathrm{Fac}(w) \subseteq \mathrm{Fac}(D_\varphi)\} = \{0,1\}^\omega \setminus \{0,1\}^* 11 \{0,1\}^\omega.$$

Straightforward but crucial observation: $\mathrm{Fac}(S_\varphi) = \mathcal{N}_F$.

Representing integers via positional numeration systems U

Let $U = (U(i))_{i\geq 0}$ be an increasing integer sequence such that $U(0) = 1$ and

$$C_U := \sup\{i \geq 0 : \lceil \tfrac{U(i+1)}{U(i)} \rceil\} < \infty.$$

We may represent any $n \in \mathbb{N}$ by using the following greedy algorithm.

First, compute the least ℓ such that $n < U(\ell)$. Then for all $i = 1, \ldots, \ell$, let a_i be the greatest integer a such that

$$\sum_{j=1}^{i-1} a_j U(\ell - j) + aU(\ell - i) \leq n.$$

We get that

$$\sum_{i=1}^{\ell} a_i U(\ell - i) = n.$$

The finite word $\mathrm{rep}_U(n) = a_1 \cdots a_\ell$ is called the U-expansion of n.

These words are written over the finite alphabet $A_U = \{0, \ldots, C_U - 1\}$.

Representing real numbers via real bases $\beta > 1$

Let $\beta > 1$ be real number (called the base).

We may represent any $x \in [0,1]$ by using the following greedy algorithm.

For all $i \geq 1$, let a_i be the greatest integer a such that

$$\sum_{j=1}^{i-1} \frac{a_j}{\beta^j} + \frac{a}{\beta^i} \leq x.$$

We get that

$$\sum_{i=1}^{\infty} \frac{a_i}{\beta^i} = x.$$

The infinite word $d_\beta(x) = a_1 a_2 \cdots$ is called the β-expansion of x.

Only finitely many digits are used, namely $0, 1, \ldots, \lfloor \beta \rfloor$.

Bertrand numeration systems

Let U be a positional numeration system.

The set $\mathcal{N}_U = 0^*\mathrm{rep}_U(\mathbb{N})$ is called the numeration language.

Two desired properties of $\mathcal{N}_U$ are:

- $\mathcal{N}_U$ is prefix-closed if all prefixes of words in $\mathcal{N}_U$ also belong to $\mathcal{N}_U$.
- $\mathcal{N}_U$ is prolongable if for all w in $\mathcal{N}_U$, the word $w0$ also belongs to $\mathcal{N}_U$.

We say that U is a Bertrand numeration system if $\mathcal{N}_U$ is both prefix-closed and prolongable.

Equivalently: $\forall w \in A_U^*,\ w \in \mathcal{N}_U \iff w0 \in \mathcal{N}_U$.

State of the art

This form of the definition of Bertrand numeration systems, as well as their names after Bertrand-Mathis, was first given in

- Bruyère & Hansel 1997. Bertrand numeration systems and recognizability.

Then it was used in

- Point 2000. On decidable extensions of Presburger arithmetic: from A. Bertrand numeration systems to Pisot numbers
- Frougny 2002. Numeration systems. (Chapter 7 of Lothaire's book "Algebraic combinatorics on words".)
- Lecomte & Rigo 2004. Real numbers having ultimately periodic representations in abstract numeration systems.
- Berthé & Rigo 2007. Odometers on regular languages.
- Charlier, Rampersad, Rigo & Waxweiler 2011. The minimal automaton recognizing $m\mathbb{N}$ in a linear numeration system.
- Massuir, Peltomäki & Rigo 2019. Automatic sequences based on Parry or Bertrand numeration systems.
- Stipulanti 2019. Convergence of Pascal-like triangles in Parry-Bertrand numeration systems.

Other works considering Bertrand numeration systems are

- Loraud 1995. β-shift, systèmes de numération et automates.
- Frougny & Solomyak 1996. On representation of integers in linear numeration systems.
- Frougny 2003. On-line digit set conversion in real base.
- Frougny, Gazeau & Krejcar 2003. Additive and multiplicative properties of point sets based on beta-integers.
- Barat, Frougny & Pethö 2005. A note on linear recurrent Mahler numbers.
- Berthé & Siegel 2007. Purely periodic β-expansions in the Pisot non-unit case.
- Frougny & Sakarovitch 2010. Number representation and finite automata. (Chapter 2 of the book "Combinatorics, automata and number theory").
- Berthé, Frougny, Rigo & Sakarovitch 2020. The carry propagation of the successor function.

The β-shift

Before giving Bertrand-Mathis's statement, we need one more notion on the real base side.

For $\beta > 1$, we let $D_\beta = \{d_\beta(x) : x \in [0,1)\}$.

The β-shift is the topological closure of D_β:

$$S_\beta = \{w \in \{0, \dots, \lceil\beta\rceil - 1\}^\omega : \mathrm{Fac}(w) \subseteq \mathrm{Fac}(D_\beta)\}.$$

Parry's characterization of elements in the β-shift

In Parry's theorem, the β-expansion and the quasi-greedy β-expansion of 1 play crucial roles.

The quasi-greedy β-expansion of 1 is

$$d_\beta^*(1) = \lim_{x \to 1^-} d_\beta(x).$$

If $d_\beta(1)$ does not end with a tail of zeros, we simply have $d_\beta^*(1) = d_\beta(1)$.

Otherwise, if $d_\beta(1) = t_1 \cdots t_n 0^\omega$ with $t_n \neq 0$, then $d_\beta^*(1) = (t_1 \cdots t_{n-1}(t_n - 1))^\omega$.

Theorem (Parry 1960)

$$S_\beta = \{w \in \{0, \dots, \lceil \beta \rceil - 1\}^\omega : \forall i \geq 1,\ w_i w_{i+1} \cdots \leq_{\text{lex}} d_\beta^*(1)\}.$$

Parry's descriptions of the 3-shift and the φ-shift

For $\beta = 3$, we get $d_3(1) = 30^\omega$ and $d_3^*(1) = 2^\omega$. So Parry's theorem gives

$$S_3 = \{w \in \{0,1,2\}^\omega : \forall i \geq 1,\ w_i w_{i+1} \cdots \leq_{\text{lex}} 2^\omega\}.$$

For $\beta = \varphi$, we get $d_\varphi(1) = 110^\omega$ and $d_\varphi^*(1) = (10)^\omega$. So Parry's theorem gives

$$S_\varphi = \{w \in \{0,1\}^\omega : \forall i \geq 1,\ w_i w_{i+1} \cdots \leq_{\text{lex}} (10)^\omega\}.$$

Bertrand-Mathis's statement

In 1989, Bertrand-Mathis stated that

$$U \text{ is Bertrand if and only if } \exists\beta > 1 \text{ such that } \mathcal{N}_U = \mathrm{Fac}(S_\beta).$$

In this case, the following hold:

a. There is a unique such β.

b. The alphabet A_U equals $\{0, \ldots, \lceil\beta\rceil - 1\}$.

c. We have

$$\forall i \geq 0, \quad U(i) = d_1 U(i-1) + d_2 U(i-2) + \cdots + d_i U(0) + 1$$

and

$$\lim_{i\to\infty} \frac{U(i)}{\beta^i} = \frac{\beta}{(\beta-1)\sum_{i=1}^{\infty} i d_i \beta^{-i}}$$

where $(d_i)_{i\geq 1} = d^*_\beta(1)$.

d. The system U has the dominant root β, i.e., $\lim_{i\to\infty} \frac{U(i+1)}{U(i)} = \beta$.

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In this case, the following hold:

a. There is a unique such β.

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c. We have

$$\forall i \geq 0, \quad U(i) = d_1 U(i-1) + d_2 U(i-2) + \cdots + d_i U(0) + 1$$

and

$$\lim_{i \to \infty} \frac{U(i)}{\beta^i} = \frac{\beta}{(\beta - 1) \sum_{i=1}^{\infty} i d_i \beta^{-i}}$$

where $(d_i)_{i \geq 1} = d^*_\beta(1)$.

d. The system U has the dominant root β, i.e., $\lim_{i \to \infty} \frac{U(i+1)}{U(i)} = \beta$.

Full characterization of Bertrand numeration systems

For a real number $\beta > 1$, define

$$S'_\beta = \{w \in \{0, \dots, \lfloor\beta\rfloor\}^\omega : \forall i \geq 1,\ w_i w_{i+1} \cdots \leq_{\text{lex}} d_\beta(1)\}.$$

Theorem (Charlier, Cisternino & Stipulanti 2022)

A positional numeration system U is Bertrand if and only if one of the following occurs.

1. *For all $i \geq 0$, $U(i) = i + 1$.*
2. *There exists a real number $\beta > 1$ such that $\mathcal{N}_U = \mathrm{Fac}(S_\beta)$.*
3. *There exists a real number $\beta > 1$ such that $\mathcal{N}_U = \mathrm{Fac}(S'_\beta)$.*

Moreover, in Case 2 (resp. Case 3), the following hold:

a. *There is a unique such β.*

b. *The alphabet A_U equals $\{0, \ldots, \lceil\beta\rceil - 1\}$ (resp. $\{0, \ldots, \lfloor\beta\rfloor\}$).*

c. *We have*

$$\forall i \geq 0, \quad U(i) = a_1 U(i-1) + a_2 U(i-2) + \cdots + a_i U(0) + 1$$

and

$$\lim_{i\to\infty} \frac{U(i)}{\beta^i} = \frac{\beta}{(\beta - 1)\sum_{i=1}^{\infty} i a_i \beta^{-i}}$$

where $(a_i)_{i\geq 1}$ is $d^*_\beta(1)$ (resp. $d_\beta(1)$).

d. *The system U has the dominant root β, i.e., $\lim_{i\to\infty} \frac{U(i+1)}{U(i)} = \beta$.*

Non-canonical Bertrand systems and non-canonical β-shifts

Let β be a simple Parry number, i.e., such that $d_\beta(1)$ ends with a tail of zeroes. In this case, $d^*_\beta(1) \neq d_\beta(1)$, and hence there are two Bertrand numeration systems associated with β.

- The canonical Bertrand system is built from the digits of $d^*_\beta(1)$.
- The non-canonical Bertrand system is built from the digits of $d_\beta(1)$.

Similarly,

- The set $S_\beta = \{w \in \{0,\ldots,\lceil\beta\rceil - 1\}^\omega : \forall i \geq 1,\ w_i w_{i+1} \cdots \leq_{\text{lex}} d^*_\beta(1)\}$ is called the canonical β-shift
- The set $S'_\beta = \{w \in \{0,\ldots,\lfloor\beta\rfloor\}^\omega : \forall i \geq 1,\ w_i w_{i+1} \cdots \leq_{\text{lex}} d_\beta(1)\}$ is called the non-canonical β-shift.

Canonical Bertrand numeration system associated with 3

Since $d^*_\beta(1) = 2^\omega$, the canonical Bertrand system associated with 3 is given by

$$\forall i \geq 0,\ \ U(i) = 2U(i-1) + 2U(i-2) + \cdots + 2U(0) + 1.$$

Thus, $U(0) = 1$ and for all $i \geq 0$, one has

$$\begin{aligned} U(i+1) - U(i) &= (2U(i) + 2U(i-1) + \cdots + 2U(0) + 1) \\ &\qquad - (2U(i-1) + 2U(i-2) + \cdots + 2U(0) + 1) \\ &= 2U(i). \end{aligned}$$

Hence $U(i+1) = 3U(i)$ for all $i \geq 0$.

We see that this is precisely the integer base 3 numeration system $U = (3^i)_{i \geq 0}$.

Non-canonical Bertrand system associated with 3

Since $d_3(1) = 30^\omega$, the non-canonical Bertrand system associated with 3 is given by

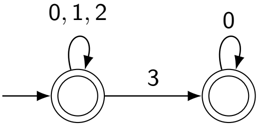
$$\forall i \geq 0,\ \ U(i) = 3U(i-1) + 1.$$

We have $U = (1, 4, 13, 40, 121, \dots)$.

The corresponding numeration language $\mathcal{N}_U$ is equal to $\mathrm{Fac}(S_3')$ where the non-canonical 3-shift is

$$S_3' = \{w \in \{0,1,2,3\}^\omega : \forall i \geq 1,\ \ w_i w_{i+1} \cdots \leq_{\mathrm{lex}} 30^\omega\}.$$

It is accepted by the DFA



From this DFA, we can see that U is Bertrand, i.e., that $\mathcal{N}_U$ is prefix-closed and prolongable.

Canonical Bertrand system associated with φ

Since $d^*_\varphi(1) = (10)^\omega$, the canonical Bertrand system associated with φ is given by

$$\forall i \geq 0,\ \ U(i) = \begin{cases} U(i-1) + U(i-3) + \cdots + U(1) + 1, & \text{if } i \text{ is even;} \\ U(i-1) + U(i-3) + \cdots + U(0) + 1, & \text{if } i \text{ is odd.} \end{cases}$$

Thus, $U(0) = 1$, $U(1) = U(0) + 1 = 2$ and for all $i \geq 0$, one has

$$U(i+2) - U(i) = U(i+1).$$

Hence $U(i+2) = U(i+1) + U(i)$ for all $i \geq 0$.

We see that this is precisely the Zeckendorf system $F = (1, 2, 3, 5, 8, 13, \ldots)$.

Non-canonical Bertrand system associated with φ

Since $d_\varphi(1) = 110^\omega$, the non-canonical Bertrand system associated with φ is given by

$$\forall i \geq 0,\ \ U(i) = U(i-1) + U(i-2) + 1,$$

i.e.,

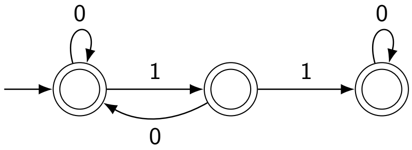
$$U(0) = 1,\ \ U(1) = U(0) + 1 = 2,\ \text{ and } \forall i \geq 0,\ \ U(i+2) = U(i+1) + U(i) + 1.$$

We have $U = (1, 2, 4, 7, 12, 20, 33, 54, \ldots)$.

The corresponding numeration language $\mathcal{N}_U$ is equal to $\mathrm{Fac}(S'_\varphi)$ where the non-canonical φ-shift is

$$S'_\varphi = \{w \in \{0,1\}^\omega : \forall i \geq 1,\ \ w_i w_{i+1} \cdots \leq_{\mathrm{lex}} 110^\omega\}.$$

It is accepted by the DFA



From this DFA, we can check that U is indeed a Bertrand numeration system.

Intermediate β-representations of 1

At first, our guess was that there could be other kinds of Bertrand numeration systems, namely any U defined by

$$\forall i \geq 0, \quad U(i) = a_1 U(i-1) + a_2 U(i-2) + \cdots + a_i U(0) + 1$$

with the sequence of coefficients given by

$$(a_i)_{i \geq 1} = (t_1 \cdots t_{n-1}(t_n - 1))^k t_1 \cdots t_n 0^\omega$$

for any $k \in \mathbb{N} \cup \{\infty\}$.

In fact, what we get is that only the cases $k = 0$ or $k = \infty$ are possible.

Intermediates are not Bertrand

Let $(a_i)_{i\geq 1} = 230^\omega$. We have $\frac{2}{3} + \frac{3}{3^2} = 1$.

Define U by

$$\begin{aligned} U(0) &= 1, \\ U(1) &= 2U(0) + 1 = 3, \\ U(i) &= 2U(i-1) + 3U(i-2) + 1, \quad i \geq 2. \end{aligned}$$

We get $U = (1, 3, 10, 30, 91, \ldots)$.

This system is not Bertrand since for example, $30 \in \mathcal{N}_U$ but $3, 300 \notin \mathcal{N}_U$, showing that $\mathcal{N}_U$ is neither prefix-closed nor prolongable.

In fact, we have

$$U(i+1) = \begin{cases} 3U(i), & \text{if } i \text{ is odd;} \\ 3U(i) + 1, & \text{if } i \text{ is even.} \end{cases}$$

Intermediates are not Bertrand

Let $(a_i)_{i\geq 1} = 10110^\omega$. We have $\frac{1}{\varphi} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} = 1$.

Define U by

$$\begin{aligned}
U(0) &= 1, \\
U(1) &= U(0) + 1 = 2, \\
U(2) &= U(1) + 1 = 3, \\
U(3) &= U(2) + U(0) + 1 = 5, \\
U(i) &= U(i-1) + U(i-3) + U(i-4) + 1, \quad i \geq 4.
\end{aligned}$$

We get $U = (1, 2, 3, 5, 9, 15, 24, 39, \ldots)$.

This system is not Bertrand since for example, $1100, 11000 \in \mathcal{N}_U$ but $11, 110, 110000 \notin \mathcal{N}_U$, showing that $\mathcal{N}_U$ is neither prefix-closed nor prolongable.

In fact, we have

$$U(i+2) = \begin{cases} U(i+1) + U(i), & \text{if } i \equiv 2, 3 \pmod 4; \\ U(i+1) + U(i) + 1, & \text{if } i \equiv 0, 1 \pmod 4. \end{cases}$$

Proposition (Hollander 1998)

Let U be a positional numeration system such that $\lim_{i\to\infty} \frac{U(i+1)}{U(i)} = \beta > 1$.

- *If β is not a simple Parry number, then*

$$\lim_{i\to\infty} \mathrm{rep}_U(U(i)-1) = d_\beta(1).$$

- *If $d_\beta(1) = t_1 \cdots t_n$ with $t_n \neq 0$, then for all $\ell \geq 0$, there exists $I \geq 0$ such that for all $i \geq I$, there exists $k \geq 0$ such that*

$$\mathrm{Pref}_\ell(\mathrm{rep}_U(U(i)-1)) = \mathrm{Pref}_\ell((t_1 \cdots t_{n-1}(t_n-1))^k t_1 \cdots t_n 0^\omega).$$

Proposition (Charlier, Cisternino & Stipulanti 2022)

Let U be a positional numeration system such that $\lim_{i\to\infty} \frac{U(i+1)}{U(i)} = \beta > 1$.

If $\lim_{i\to\infty} \mathrm{rep}_U(U(i)-1)$ exists, then it is either $d^*_\beta(1)$ or $d_\beta(1)$.

Another characterization of Bertrand numeration systems

Theorem (Charlier, Cisternino & Stipulanti 2022)

A positional numeration system U is Bertrand if and only if one of the following conditions is satisfied.

1. *We have* $\mathrm{rep}_U(U(i)-1) = \mathrm{Pref}_i(10^\omega)$ *for all* $i \geq 0$.
2. *There exists* $\beta > 1$ *such that* $\mathrm{rep}_U(U(i)-1) = \mathrm{Pref}_i(d_\beta^*(1))$ *for all* $i \geq 0$.
3. *There exists* $\beta > 1$ *such that* $\mathrm{rep}_U(U(i)-1) = \mathrm{Pref}_i(d_\beta(1))$ *for all* $i \geq 0$.

Understanding the non-canonical β-shift

A subshift (i.e., a subset of A^ω that is topologically closed and shift-invariant) is said to be sofic if its factors form a language that is accepted by a finite automaton.

A Parry number is a real number $\beta > 1$ such that $d_\beta(1)$ is ultimately periodic (or equivalently, $d^*_\beta(1)$ is ultimately periodic).

Theorem (Bertrand-Mathis 1986)

For $\beta > 1$, the subshift S_β is sofic if and only if β is a Parry number.

We get the analogous result:

Proposition

For $\beta > 1$, the subshift S'_β is sofic if and only if β is a Parry number.

The entropy of a subshift S of A^ω is

$$\lim_{i\to\infty} \frac{1}{i} \log(\mathrm{Card}(\mathrm{Fac}(S) \cap A^i)).$$

Theorem

For all $\beta > 1$*, the* β*-shift* S_β *has entropy* $\log(\beta)$.

We have the analogous result:

Proposition

For all $\beta > 1$*, the subshift* S'_β *has entropy* $\log(\beta)$.

Some negative results

A subshift S is said to be of finite type if there exists a finite set $X \subset A^*$ such that $S = \{w \in A^{\mathbb{N}} : \mathrm{Fac}(w) \cap X = \emptyset\}$.

Theorem

For all $\beta > 1$, the β-shift S_β is of finite type if and only is β is a simple Parry number.

However:

Proposition

For any simple Parry number $\beta > 1$, the subshift S'_β is not of finite type.

A subshift S is said to be coded if there exists a prefix code $Y \subset A^*$ such that $\mathrm{Fac}(S) = \mathrm{Fac}(Y^*)$.

Theorem

For all $\beta > 1$, the canonical β-shift S_β is coded.

In order to show that S'_β is not coded, we prove the stronger statement that S'_β is not irreducible.

A subshift S is said to be irreducible if for all $u, v \in \mathrm{Fac}(S)$, there exists $w \in \mathrm{Fac}(S)$ such that $uwv \in \mathrm{Fac}(S)$.

Proposition

For any simple Parry number β, the non-canonical β-shift S'_β is not irreducible.

A relation between the number of words of length i in the canonical and the non-canonical β-shifts.

Suppose that $\beta > 1$ is a real number such that $d_\beta(1) = t_1 \cdots t_n 0^\omega$ with $n \geq 1$ and $t_n \neq 0$, and let U and U' respectively be the canonical and non-canonical Bertrand numeration systems associated with β.

Thanks to our characterization of Bertrand systems, we know that for all $i \geq 0$,

- the number of words of length i in $\mathrm{Fac}(S_\beta)$ is $U(i)$
- the number of words of length i in $\mathrm{Fac}(S'_\beta)$ is $U'(i)$.

Proposition

For all $i \geq 0$, one has $U'(i+n) = U(i+n) + U'(i)$.

$U'(i+n) = U(i+n) + U'(i)$ for all $i \geq 0$

For $\beta = 3$, we have $d_3(1) = 30^\omega$, hence $n = 1$.

We have seen that

$$U(i) = 3^i \quad \forall i \geq 0$$

and that

$$U'(0) = 1, \quad U'(i+1) = 3U'(i) + 1 \quad \forall i \geq 0.$$

i	0	1	2	3	4	5	$\cdots$
$U(i)$	1	3	9	27	81	243	
$U'(i)$	1	4	13	40	121	364	

$U'(i+n) = U(i+n) + U'(i)$ for all $i \geq 0$

For $\beta = \varphi$, we have $d_\varphi(1) = 110^\omega$, hence $n = 2$.

We have seen that

$$U(0) = 1,\ U(1) = 2, \quad U(i+2) = U(i+1) + U(i) \quad \forall i \geq 0$$

and that

$$U'(0) = 1,\ U'(1) = 2, \quad U'(i+2) = U(i+1) + U(i) + 1 \quad \forall i \geq 0.$$

i	0	1	2	3	4	5	6	7	8	$\cdots$
$U(i)$	1	2	3	5	8	13	21	34	55	
$U'(i)$	1	2	4	7	12	20	33	54	88	

Thank you!

Merci !